Induced corepresentations of locally compact quantum groups

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We introduce the construction of induced corepresentations in the setting of locally compact quantum groups and prove that the resulting induced corepresentations are unitary under some mild integrability condition. We also establish a quantum analogue of the classical bijective correspondence between quasi-invariant measures and certain measures on the larger locally compact group.

Introduction

Consider a closed subgroup H of a locally compact group G together with a strongly continuous unitary representation u of H on a Hilbert space K. The construction of the induced representation uses these three ingredients to supply a new strongly continuous unitary representation ρ of the larger group G on a new Hilbert space $\mathcal{K}$.

Induced group representations for finite groups where first introduced by Frobenius in 1898. The theory of induced representations for locally compact groups was initiated by Mackey in 1949 (see [14], [15] and [16]). He restricted himself to locally compact groups satisfying the second axiom of countability and separable Hilbert spaces. The general case was treated by Blattner in 1961 (see [3]).

The role of the theory of induced representations in the classical theory of locally compact groups can hardly be underestimated. This construction is for instance one of the primary instruments to construct the different series of special representations of various non-compact locally compact groups.

Building on the work of Enock & Schwartz and Kac & Vainerman in the theory of Kac algebras (see [5]), Baaj & Skandalis, Kirchberg, Woronowicz and Masuda & Nakagami in the more general setting of quantum groups, S. Vaes and the author developed a relatively simple definition of a locally compact quantum group. These locally compact quantum groups can present themselves in different forms. Two of these forms are formulated in an C*-algebraic setting (see [11] and [10]) but in this paper we are mainly in the von Neumann algebraic setting (see [12]).

Building on this definition, it is important to further develop the proper theory of locally compact quantum groups and in this way obtain a theory that is as rich as the classical theory of locally compact groups. In this light, it is very natural to generalize the theory of induced representations to the quantum

group setting. This paper deals with the first step in this generalization, the construction of the induced corepresentation and establishing its unitarity.

Starting from two von Neumann algebraic quantum groups (M, Δ) and (N, Δ) together with a special action $\alpha : M \to M \otimes N$ (that mimics the role of the obvious action of the group H on G) and a unitary corepresentation U of the 'smaller' quantum group (N, Δ), we construct a new corepresentation ρ of (M, Δ). Under a mild integrability condition, we prove that this new corepresentation ρ is again unitary. The difficulty of proving the unitarity of the corepresentation ρ is even more pronounced than proving the unitarity of multiplicative unitaries popping up in the axiomatic approach to quantum groups.

The further development of the theory of induced corepresentations will be postponed to later papers.

Thanks to the theory of normal faithful semifinite weights on von Neumann algebras, the existence of the quantum version of quasi-invariant measures in the classical theory becomes a mere observation in the quantum setting. Although the classical quasi-invariant measures provide Radon Nikodym derivatives, the role of these Radon Nikodym derivatives is much less important in the quantum setting because they seize to exist in general. Since these Radon Nikodym derivatives appear prominently in the definition of the induced representations in the classical theory, another approach had to be found.

In this respect, the results in [19] come to the rescue and play a vital role in getting the theory going. In this paper, S. Vaes proves that every action of a von Neumann algebraic quantum group on another von Neumann algebra is implemented by a canonical unitary corepresentation (this is a generalization of a classical result of U. Haagerup).

The paper is organized as follows. In section 1 we give a short overview of weight theory on von Neumman algebras. The second section contains the definition of a von Neumann algebraic quantum group. We fix the data that we will need for the rest of the paper in section 3. The carrier space of the induced corepresentation is constructed in section 4, together with a useful characterization of the tensor product of this carrier space with any Hilbert space. The induced corepresentation itself is constructed in section 5. Under an integrability condition, we provide an important dense subspace of the carrier space in section 6. In section 7, we prove the unitarity of the induced corepresentation under this integrability condition. In the last section we establish a bijective correspondence between the quasi-invariant weights and a certain class of weights on the 'larger' von Neumann algebra M.

We end this introduction with some notations and conventions. If V is a normed space and L is a subset of V, then $\langle L \rangle$ will denote the linear span of L, $[L]$ will denote the closed linear span of L. For any set I, we define $F(I)$ to be the set of all finite subset of I and we turn $F(I)$ into a directed set through the inclusion relation. The identity map will be denoted by ι. If V, W are two vector spaces, the algebraic tensor product is denoted by $V \odot W$ (we will also use $\odot$ for the tensor product of linear maps).

Consider two von Neumann algebras M and N. The von Neumann algebraic tensor product of M and N will be denoted by $M \otimes N$ (again, we will also use $\otimes$ for the tensor product of sufficiently nice normal linear maps). We denote the flip automorphism from $M \otimes N$ to $N \otimes M$ by χ.

1. Preliminaries on weights and operator valued weights

Before starting our discussion on weights, we would first like to mention the following useful property. Consider Hilbert spaces H,K and elements $X, Y \in B(H \otimes K)$, $v, w \in K$ and $(e_i)_{i \in I}$ an othonormal basis for K. Then the net

$$\Big(\sum_{i \in J} (\iota \otimes \omega_{e_i, w})(X) (\iota \otimes \omega_{v, e_i})(Y) \Big)_{J \in F(I)} \tag{1.1}$$

is bounded and converges strongly* to $(\iota \otimes \omega_{v,w})(XY)$ (see e.g. lemma 9.5 of [11] for a proof).

We assume that the reader is familiar with the theory of normal faithful semi-finite weights (in short, n.s.f. weights) on von Neumann algebras. Nevertheless, let us fix some notations. So let φ be a n.f.s. weight on a von Neumann algebra. Then we define the following sets:

1. $\mathcal{M}_\varphi^+ = \{\, x \in M^+ \mid \varphi(x) < \infty \,\}$, so $\mathcal{M}_\varphi^+$ is a hereditary cone in M^+,
2. $\mathcal{N}_\varphi = \{\, x \in M \mid x^* x \in \mathcal{M}_\varphi^+ \,\}$, so $\mathcal{N}_\varphi$ is a left ideal in M,
3. $\mathcal{M}_\varphi = $ the linear span of $\mathcal{M}_\varphi^+$ in M, so $\mathcal{M}_\varphi$ is a sub *-algebra of M.

There exists a unique linear map $F : \mathcal{M}_\varphi \to \mathbb{C}$ such that $F(x) = \varphi(x)$ for all $x \in \mathcal{M}_\varphi^+$. For all $x \in \mathcal{M}_\varphi$, we set $\varphi(x) = F(x)$.

A GNS-construction for φ is a triple $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$, where H_φ is a Hilbert space, $\pi_\varphi : M \to B(H_\varphi)$ is a normal *-homomorphism and $\Lambda_\varphi : \mathcal{N}_\varphi \to H_\varphi$ is a σ-strong* closed linear map with dense range such that

1. $\langle \Lambda_\varphi(x), \Lambda_\varphi(y) \rangle = \varphi(y^* x)$ for all $x, y \in \mathcal{N}_\varphi$,
2. $\Lambda_\varphi(x\,y) = \pi_\varphi(x) \Lambda_\varphi(y)$ for all $x \in M$ and $y \in \mathcal{N}_\varphi$.

Such a GNS-construction always exist. The modular group of φ will be denoted by σ^φ, the modular operator by ∇_φ and the modular conjugation by J_φ (these last two objects are defined with respect to the GNS-construction). Recall that $\Lambda_\varphi(\mathcal{N}_\varphi^* \cap \mathcal{N}_\varphi)$ is a core for the operator $J_\varphi \nabla_\varphi^{\frac{1}{2}}$ and $(J_\varphi \nabla_\varphi^{\frac{1}{2}}) \Lambda_\varphi(x) = \Lambda_\varphi(x^*)$ for all $x \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$.

Consider two von Neumann algebras M, N. Let φ be a n.f.s. weight on M with GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ and let ψ be a n.s.f. weight on N with GNS-construction $(H_\psi, \pi_\psi, \Lambda_\psi)$. The tensor product weight $\varphi \otimes \psi$ is a n.s.f. weight on $M \otimes N$ (see e.g. definition 8.2 of [17] for a definition). This tensor product weight has a GNS-construction $(H_\varphi \otimes H_\psi, \pi_\varphi \otimes \pi_\psi, \Lambda_\varphi \otimes \Lambda_\psi)$ where $\Lambda_\varphi \otimes \Lambda_\psi : \mathcal{N}_{\varphi \otimes \psi} \to H_\varphi \otimes H_\psi$ is the σ-strong* closure of $\Lambda_\varphi \odot \Lambda_\psi : \mathcal{N}_\varphi \odot \mathcal{N}_\psi \to H_\varphi \otimes H_\psi$.

Let M be any von Neumann algebra. For the definition of the extended positive part M_{ext}^+ of M we refer to definition 1.1 of [6]. For $T \in M_{\text{ext}}^+$ and $\omega \in M_*^+$, we set $\omega(T) = T(\omega) \in [0, \infty]$. Recall that there exists an embedding $M^+ \hookrightarrow M_{\text{ext}}^+ : x \mapsto x^\sharp$ such that $x^\sharp(\omega) = \omega(x)$ for all $x \in M^+$ and $\omega \in M_*^+$. We will use this embedding to identify M^+ as a subset of M_{ext}^+.

Consider a von Neumann algebra M and a sub von Neumann algebra N of M. The definition of an operator valued weight from M to N is given in definition 2.1 of [6].

Now consider two von Neumann algebras M and N and a n.f.s. weight φ on M with GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$. We identify N with $1 \otimes N$ as a sub von Neumann algebra of $M \otimes N$ to get into the framework of operator valued weights. The operator valued weight $\varphi \otimes \iota : (M \otimes N)^+ \to N_{\text{ext}}^+$ is defined in such a way that for $x \in (M \otimes N)^+$, we have that

$$\omega\big((\varphi \otimes \iota)(x)\big) = \varphi\big((\iota \otimes \omega)(x)\big) \ .$$

As for weights we define the following sets:

1. $\mathcal{M}_{\varphi \otimes \iota}^+ = \{\, x \in (M \otimes N)^+ \mid (\varphi \otimes \iota)(x) \in N^+ \,\}$, so $\mathcal{M}_{\varphi \otimes \iota}^+$ is a hereditary cone of $(M \otimes N)^+$,
2. $\mathcal{N}_{\varphi \otimes \iota} = \{\, x \in M \otimes N \mid x^* x \in \mathcal{M}_{\varphi \otimes \iota}^+ \,\}$, so $\mathcal{N}_{\varphi \otimes \iota}$ is a left ideal in $M \otimes N$,

3. $\mathcal{M}_{\varphi\otimes\iota}$ = the linear span of $\mathcal{M}_{\varphi\otimes\iota}^+$ in $M \otimes N$, so $\mathcal{M}_{\varphi\otimes\iota}$ is a sub *-algebra of $M \otimes N$.

There exists a unique linear map $G : \mathcal{M}_{\varphi\otimes\iota} \to N$ such that $G(x) = (\varphi \otimes \iota)(x)$ for all $x \in \mathcal{M}_{\varphi\otimes\iota}^+$. For all $x \in \mathcal{M}_{\varphi\otimes\iota}$, we set $(\varphi \otimes \iota)(x) = G(x)$. Let $a \in \mathcal{M}_\varphi$ and $b \in N$. Then it is easy to see that $a \otimes b$ belongs to $\mathcal{M}_{\varphi\otimes\iota}$ and $(\varphi \otimes \iota)(a \otimes b) = \varphi(a)\, b$.

Thanks to the remark after lemma 1.4 of [6], we also have the following characterization of $\mathcal{M}_{\varphi\otimes\iota}^+$: Let $x \in (M \otimes N)^+$, then x belongs to $\mathcal{M}_{\varphi\otimes\iota}^+ \Leftrightarrow \varphi((\iota \otimes \omega)(x)) < \infty$ for all $\omega \in M_*^+$.

Let $x \in \mathcal{N}_{\varphi\otimes\iota}$ and $\omega \in N_*$. The inequality $(\iota \otimes \omega)(x)^*(\iota \otimes \omega)(x) \leq \|\omega\|\,(\iota \otimes |\omega|)(x^*x)$ implies that $(\iota \otimes \omega)(x) \in \mathcal{N}_\varphi$ and

$$\|\Lambda_\varphi((\iota \otimes \omega)(x))\| \leq \|\omega\|\, \|(\Lambda_\varphi \otimes \iota)(x)\| \ .$$

As for weights, there exists also a KSGNS-construction for this operator valued weight $\varphi \otimes \iota$. For this, let H denote the Hilbert space on which N acts.

Definition 1.1. *There exists a unique linear map $\Lambda_\varphi \otimes \iota : \mathcal{N}_{\varphi\otimes\iota} \to B(H, H_\varphi \otimes H)$ such that $\langle(\Lambda_\varphi \otimes \iota)(x)\,v, \Lambda_\varphi(a) \otimes w\rangle = \langle(\varphi \otimes \iota)((a^* \otimes 1)x)\,v, w\rangle$ for all $x \in \mathcal{N}_{\varphi\otimes\iota}$, $a \in \mathcal{N}_\varphi$ and $v, w \in H$. Moreover, the following properties hold:*

1. $(\Lambda_\varphi \otimes \iota)(x\,y) = (\pi_\varphi \otimes \iota)(x)\,(\Lambda_\varphi \otimes \iota)(y)$ *for all $x \in M \otimes N$ and $y \in \mathcal{N}_{\varphi\otimes\iota}$*
2. $(\Lambda_\varphi \otimes \iota)(y)^*(\Lambda_\varphi \otimes \iota)(x) = (\varphi \otimes \iota)(y^*x)$ *for all $x, y \in \mathcal{N}_{\varphi\otimes\iota}$.*
3. $H_\varphi \otimes H = [\,(\Lambda_\varphi \otimes \iota)(x)\,v \mid x \in \mathcal{N}_{\varphi\otimes\iota}, v \in H\,]$.

We have of course that $(\Lambda_\varphi \otimes \iota)(a \otimes b)\,v = \Lambda_\varphi(a) \otimes b\,v$ for all $a \in \mathcal{N}_\varphi$, $b \in N$ and $v \in H$.

The proof of the existence of the map $\Lambda_\varphi \otimes \iota$ and most of its properties can be extracted from the following result:

Result 1.2. *Consider $x \in \mathcal{N}_{\varphi\otimes\iota}$, $v \in H$ and an orthonormal basis $(e_i)_{i\in I}$ for H. Then $\sum_{i\in I}\|\Lambda_\varphi((\iota \otimes \omega_{v,e_i})(x))\|^2 < \infty$ and*

$$(\Lambda_\varphi \otimes \iota)(x)\,v = \sum_{i\in I} \Lambda_\varphi((\iota \otimes \omega_{v,e_i})(x)) \otimes e_i \ .$$

It is also possible to show that $\Lambda_\varphi \otimes \iota : \mathcal{N}_{\varphi\otimes\iota} \to B(H, H_\varphi \otimes H)$ is closed for the σ-strong* topology on $M \otimes N$ and the strong topology on $B(H, H_\varphi \otimes H)$ (see proposition 3.23 of [13] for a similar result in the C*-algebra setting). One can even prove the following fact:

Let $x \in \mathcal{N}_{\varphi\otimes\iota}$. Then their exists a net $(x_i)_{i\in I}$ in $\mathcal{N}_\varphi \odot N$ such that $\|x_i\| \leq \|x\|$ and $\|(\Lambda_\varphi \otimes \iota)(x_i)\| \leq \|(\Lambda_\varphi \otimes \iota)(x)\|$ for all $i \in I$, $(x_i)_{i\in I}$ converges strongly* to x and $\big((\Lambda_\varphi \otimes \iota)(x_i)\big)_{i\in I}$ converges strongly* to $(\Lambda_\varphi \otimes \iota)(x)$ (techniques for the proof of this fact can be found in the proofs of proposition 3.28 of [13] and proposition 7.9 of [9]).

We should also mention the following dominated convergence property (see also proposition 3.24 of [13]): Let $(x_i)_{i\in I}$ be a net in $\mathcal{M}_{\varphi\otimes\iota}^+$ and x an element in $\mathcal{M}_{\varphi\otimes\iota}$ such that $x_i \leq x$ for all $i \in I$ and $(x_i)_{i\in I}$ converges strongly to x. Then $\big((\varphi \otimes \iota)(x_i)\big)_{i\in I}$ converges strongly to $(\varphi \otimes \iota)(x)$.

It is clear that we also define $\iota \otimes \varphi$ and $\iota \otimes \Lambda_\varphi$ in a similar way. But we will also use the following variations. Let L, M, N be three von Neumann algebras and let φ be a n.s.f. weight on L with GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$. Then we define $\varphi \otimes \iota_M \otimes \iota_N = \varphi \otimes \iota_{M\otimes N}$ and $\Lambda_\varphi \otimes \iota_M \otimes \iota_N = \Lambda_\varphi \otimes \iota_{M\otimes N}$. Similarly for $\iota_M \otimes \iota_N \otimes \varphi$.

In order to define $\iota_M \otimes \varphi \otimes \iota_N$, we use a permutation operator. Therefore let $\chi : M \otimes L \to L \otimes M$ denote the flip *-isomorphism, then we set $\iota_M \otimes \varphi \otimes \iota_N = (\varphi \otimes \iota_M \otimes \iota_N)(\chi \otimes \iota_N)$. If M acts on H and N acts on K, we define $\iota_M \otimes \Lambda_\varphi \otimes \iota_N = (\Sigma \otimes 1_K)(\Lambda_\varphi \otimes \iota_M \otimes \iota_N)(\chi \otimes \iota_N)$ where $\Sigma : H_\varphi \otimes H \to H \otimes H_\varphi$ denotes the flip transformation.

2. The definition of a von Neumann algebraic quantum group

In [11] we defined reduced C*-algebraic quantum groups. But it is also possible to formulate the theory in the von Neumann algebra setting as was done in [12]. In this paper we will use this alternative approach because it better suits our needs. It should be said however that there exists a natural bijection between these two notions of quantum groups in the operator algebra setting.

Definition 2.1. *Consider a von Neumann algebra and a unital normal *-homomorphism $\Delta : M \to M \otimes M$ such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. Assume moreover the existence of n.s.f. weights φ and ψ on M such that*

1. *We have for all $a \in \mathcal{M}_\varphi^+$ and $\omega \in M_*^+$ that $\varphi((\omega \otimes \iota)\Delta(a)) = \varphi(a)\,\omega(1)$.*
2. *We have for all $a \in \mathcal{M}_\psi^+$ and $\omega \in M_*^+$ that $\psi((\iota \otimes \omega)\Delta(a)) = \psi(a)\,\omega(1)$.*

Then we call the pair (M, Δ) a von Neumann algebraic quantum group.

An elaborate discussion about von Neumann algebraic quantum groups and their relation with their C*-algebraic counterparts can be found in [12].

Let us fix some further notations and terminology. Therefore consider a von Neumann algebraic quantum group (M, Δ). Fix also a n.s.f. weight φ on M satisfying property (1) of definition 2.1 (such a weight is called a left Haar weight of (M, Δ)).

There exists a unique σ-strongly* closed mapping S in M such that

1. We have for all $a, b \in \mathcal{N}_\varphi$ that $(\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b)) \in D(S)$ and $S\big((\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b))\big) = (\iota \otimes \varphi)((1 \otimes a^*)\Delta(b))$
2. The space $\langle\, (\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b)) \mid a, b \in \mathcal{N}_\varphi \,\rangle$ is a σ-strong* core for S.

There exist a unique anti *-automorphism R on M and a unique σ-strongly* continuous one parameter group τ on M such that

$$R^2 = \iota \qquad\qquad R\tau_t = \tau_t R \quad \text{ for all } t \in \mathbb{R} \qquad\qquad S = R\tau_{-\frac{i}{2}} \ .$$

We call S the antipode, R the unitary antipode and τ the scaling group of our quantum group (M, Δ). There exists a positive number $\nu > 0$ satisfying $\varphi\,\tau_t = \nu^{-t}\,\varphi$ for all $t \in \mathbb{R}$. The number ν is referred to as the scaling constant of (M, Δ).

Since $\chi(R \otimes R)\Delta = \Delta R$, the n.s.f. weight $\psi = \varphi R$ satisfies condition (2) of definition 2.1. We also get that $\psi\,\sigma_t^\varphi = \nu^{-t}\,\psi$ for $t \in \mathbb{R}$. Hence there exists a positive self-adjoint operator δ affiliated to M such that $\sigma_t^\varphi(\delta) = \nu^t\,\delta$ for all $t \in \mathbb{R}$ and $\psi = \varphi_\delta$ (for a precise definition of φ_δ, see definition 1.3 of [18]). In other words, ψ is absolutely continuous with respect to φ and δ is the Radon Nykodim derivative of ψ with respect to φ. The element δ is called the modular element of (M, Δ).

3. Fixing the data

In this section we will introduce the data that we will use throughout this paper. So let (M, Δ_M) and (N, Δ_N) be two von Neumann algebraic quantum groups with left Haar weights φ_M and φ_N respectively. Without loss of generality, we may and will assume that M and N are in standard form with respect to Hilbert spaces H_M and H_N respectively. We also fix GNS-constructions (H_M, ι, Λ_M) and (H_N, ι, Λ_N) for φ_M and φ_N respectively. Throughout this paper, the natural objects (like the antipode, the unitary antipode) associated to (M, Δ_M) will get a sub- or superscript M. The same rule applies to (N, Δ_N).

Define the right Haar weights ψ_M and ψ_N on (M, Δ_M) and (N, Δ_N) respectively as $\psi_M = \varphi_M\,R_M$ and $\psi_N = \varphi_N\,R_N$, where R_M and R_N are the unitary antipodes of (M, Δ_M) and (N, Δ_N) respectively.

Let δ_M and δ_N denote the modular elements of (M, Δ_M) and (N, Δ_N) respectively. So δ_M is the Radon Nykodim derivative of ψ_M with respect to φ_M and δ_N is the Radon Nykodim derivative of ψ_N with respect to φ_N. We define the GNS-constructions (H_M, ι, Γ_M) and (H_N, ι, Γ_N) of ψ_M and ψ_N respectively by setting $\Gamma_M = (\Lambda_M)_{\delta_M}$ and $\Gamma_N = (\Lambda_N)_{\delta_N}$.

We also would like (N, Δ_N) to be some kind of 'sub quantum group' of (M, Δ_M). Without trying to go further into the somewhat delicate notion of a 'sub quantum group', we will impose a less restrictive condition on the pair M, N.

So we will assume the existence of a normal injective *-homomorphism $\alpha : M \to M \otimes N$ such that

$$(\alpha \otimes \iota)\alpha = (\iota \otimes \Delta_N)\alpha \qquad \text{and} \qquad (\Delta_M \otimes \iota)\alpha = (\iota \otimes \alpha)\Delta_M \ . \tag{3.2}$$

The first equality means that α is a right action of (N, Δ_N) on M. These two conditions are equivalent to the fact that there exists a morphism of quantum groups from (M, Δ_M) to (N, Δ_N), a fact that is proven in section 10 of [10].

We define the sub von Neumann algebra Q of M as

$$Q = \{ \, x \in M \mid \alpha(x) = x \otimes 1 \, \} \ .$$

This Q should be thought of as a quantum analogue of the L^∞-functions on the left coset space. The role of the quasi-invariant measure on the left coset space will be played by any normal semi finite faithful weight on Q. Therefore we fix some n.s.f. weight θ on Q together with a GNS-construction $(H_\theta, \pi_\theta, \Lambda_\theta)$ for it.

Lemma 3.1. *We have $\Delta_M(Q) \subseteq M \otimes Q$.*

Proof. Choose $x \in Q$. Take $y \in Q'$. Using equation (3.2), we get for all $\omega \in M_*$,

$$\alpha\big((\omega \otimes \iota)\Delta_M(x)\big) = (\omega \otimes \iota \otimes \iota)\big((\iota \otimes \alpha)\Delta_M(x)\big) = (\omega \otimes \iota \otimes \iota)\big((\Delta_M \otimes \iota)\alpha(x)\big) = (\omega \otimes \iota)(\Delta_M(x)) \otimes 1 \ ,$$

implying that $(\omega \otimes \iota)\Delta_M(x) \in Q$ and hence $(\omega \otimes \iota)(\Delta_M(x))\, y = y\,(\omega \otimes \iota)(\Delta_M(x))$. We conclude that $\Delta_M(x)(1 \otimes y) = (1 \otimes y)\Delta_M(x)$. Therefore $\Delta_M(x) \in (M' \otimes Q')' = M \otimes Q$. $\square$

So we see, by the coassociativity of Δ_M, that the mapping $\beta : \pi_\theta(Q) \to M \otimes \pi_\theta(Q)$, defined such that $\beta(\pi_\theta(x)) = (\iota \otimes \pi_\theta)\Delta_M(x)$ for all $x \in Q$, is a left action of (M, Δ_M) on Q. In [19], Stefaan Vaes showed that any such a left action has an implementation by a unitary corepresentation. So we define Υ to be the unitary element in $M \otimes B(H_\theta)$ such that Υ^* is the unitary implementation, as defined in definition 3.6 of [19], of the left action β. Recall from [19] that Υ enjoys the following properties:

1. $(\iota \otimes \pi_\theta)\Delta_M(x) = \Upsilon^*(1 \otimes \pi_\theta(x))\Upsilon \ $ for all $x \in Q$, $\tag{3.3}$
2. $(\Delta_M \otimes \iota)(\Upsilon) = \Upsilon_{13}\Upsilon_{23} \ . \tag{3.4}$

As a last piece of data we fix a unitary corepresentation U of (N, Δ_N) on a Hilbert space K, i.e. U is a unitary element in $N \otimes B(K)$ such that $(\Delta_N \otimes \iota)(U) = U_{13}\, U_{23}$.

4. The carrier Hilbert space of the induced corepresentation

This section is devoted to the construction of the carrier Hilbert space of our induced corepresentation. The construction of this Hilbert space is modeled on the classical case. We will also give an alternative description for the tensor products of this carrier Hilbert space with any Hilbert space.

Throughout this section we fix a Hilbert space H.

Definition 4.1. *We define the subspace $\mathcal{P}_H$ of $B(H) \otimes M \otimes B(K)$ as*

$$\mathcal{P}_H = \{\, X \in B(H) \otimes M \otimes B(K) \mid (\iota \otimes \alpha \otimes \iota)(X) = U_{34}^* X_{124} \,\} \;.$$

It is easy to check that $\mathcal{P}_H$ has the following multiplicative structure: we have for any $X \in \mathcal{P}_H$ that

1. $(Y \otimes 1) X \in \mathcal{P}_H$ for all $Y \in B(H) \otimes Q$,
2. $XY \in \mathcal{P}_H$ for all $Y \in B(H) \otimes Q \otimes B(K)$.

Also the proof of the following result is very elementary.

Lemma 4.2. *Let $X, Y \in \mathcal{P}_H$, then Y^*X belongs to $B(H) \otimes Q \otimes B(K)$.*

Proof. By definition of $\mathcal{P}_H$, we have that

$$(\iota \otimes \alpha \otimes \iota)(Y^*X) = Y_{124}^* U_{34} U_{34}^* X_{124} = (Y^*X)_{124} \;.$$

So we get for all $\omega_1 \in B(H)_*$ and $\omega_2 \in B(K)_*$ that $\alpha\big((\omega_1 \otimes \iota \otimes \omega_2)(Y^*X)\big) = (\omega_1 \otimes \iota \otimes \omega_2)(Y^*X) \otimes 1$ implying that $(\omega_1 \otimes \iota \otimes \omega_2)(Y^*X) \in Q$. Arguing as in the proof of lemma 3.1, the lemma follows. $\qquad\square$

Thanks to this lemma, we can now define a sesquilinear form $\langle\,,\rangle$ on $\mathcal{P}_H \odot (H \otimes H_\theta \otimes K)$ such that

$$\langle X \otimes v, Y \otimes w \rangle = \langle (\iota \otimes \pi_\theta \otimes \iota)(Y^*X)\, v, w \rangle$$

for all $X, Y \in \mathcal{P}_H$ and $v, w \in H \otimes H_\theta \otimes K$.

Lemma 4.3. *The inproduct $\langle\,,\rangle$ on $\mathcal{P}_H \odot (H \otimes H_\theta \otimes K)$ is positive.*

Proof. Choose $X_1, \ldots, X_n \in \mathcal{P}_H$ and $v_1, \ldots, v_n \in H \otimes H_\theta \otimes K$. Then

$$\Big\langle \sum_{i=1}^n X_i \otimes v_i, \sum_{i=1}^n X_i \otimes v_i \Big\rangle = \sum_{i,j=1}^n \langle (\iota \otimes \pi_\theta \otimes \iota)(X_j^* X_i)\, v_i, v_j \rangle \;.$$

Define $T \in M_n(B(H) \otimes Q \otimes B(K))$ such that $T_{ji} = X_j^* X_i$ for all $i, j = 1, \ldots, n$. It is clear that $T \geq 0$. Now define $S \in M_n(B(H) \otimes B(H_\theta) \otimes B(K))$ by setting $S_{ji} = (\iota \otimes \pi_\theta \otimes \iota)(T_{ji})$ for $i, j = 1, \ldots, n$, then it is clear that $S \geq 0$. Thus,

$$\Big\langle \sum_{i=1}^n X_i \otimes v_i, \sum_{i=1}^n X_i \otimes v_i \Big\rangle = \sum_{i,j=1}^n \langle S_{ji}\, v_i, v_j \rangle \geq 0 \;.$$

$\qquad\square$

Set $\mathcal{J} = \{\, x \in \mathcal{P}_H \odot (H \otimes H_\theta \otimes K) \mid \langle x, x \rangle = 0 \,\}$. Then $(\mathcal{P}_H \odot (H \otimes H_\theta \otimes K))/\mathcal{J}$ carries the structure of a pre-Hilbert space in the usual way.

Notation 4.4. *We define $\mathcal{K}_H$ to be the completion of $(\mathcal{P}_H \odot (H \otimes H_\theta \otimes K))/\mathcal{J}$, the inproduct on $\mathcal{K}_H$ will also be denoted by $\langle\,,\rangle$. For $X \in \mathcal{P}_H$ and $v \in H \otimes H_\theta \otimes K$ we define the element $X \dot\otimes v$ in $\mathcal{K}_H$ as the equivalence class of $X \otimes v$ in $(\mathcal{P}_H \odot (H \otimes H_\theta \otimes K))/\mathcal{J}$.*

We only need to remember the following elementary facts about $\mathcal{K}_H$:

1. The mapping $\mathcal{P}_H \times (H \otimes H_\theta \otimes K) \to \mathcal{K}_H : (X, v) \mapsto X \dot\otimes v$ is bilinear,
2. $\langle X \dot\otimes v, Y \dot\otimes w \rangle = \langle (\iota \otimes \pi_\theta \otimes \iota)(Y^*X)\, v, w \rangle$ for all $X, Y \in \mathcal{P}_H$ and $v, w \in H \otimes H_\theta \otimes K$,

3. $\mathcal{K}_H = [\, X \mathbin{\dot{\otimes}} v \mid X \in \mathcal{P}_H, v \in H \otimes H_\theta \otimes K \,]$.

Remark 4.5. In the special case where $H = \mathbb{C}$, the Hilbert space $\mathcal{K}_{\mathbb{C}}$ will turn out to be the carrier space of the induced corepresentation. Therefore we set $\mathcal{P} = \mathcal{P}_{\mathbb{C}}$ and $\mathcal{K} = \mathcal{K}_{\mathbb{C}}$.

It is also clear that the following two properties hold

1. Let $X \in \mathcal{P}_H$ and $\omega \in B(H)_*$, then $(\omega \otimes \iota \otimes \iota)(X) \in \mathcal{P}$.
2. Let $X \in \mathcal{P}$, then $1_{B(H)} \otimes X \in \mathcal{P}_H$.

In the next proposition we will establish a natural isomorphism between $\mathcal{K}_H$ and $H \otimes \mathcal{K}$. But observe first the following basic facts. Let X be an element in $\mathcal{P}_H$. Since the map $H \otimes H_\theta \otimes K \to \mathcal{K}_H : v \mapsto X \mathbin{\dot{\otimes}} v$ is clearly bounded (with bound $\leq \|X\|$), the following holds:
Let $D \subseteq H \otimes H_\theta \otimes K$ such that $H \otimes H_\theta \otimes K = [D]$. Then $\mathcal{K}_H = [\, X \mathbin{\dot{\otimes}} v \mid X \in \mathcal{P}_H, v \in D \,]$.

Proposition 4.6. *There exists a unique unitary linear transformation $U_H : H \otimes \mathcal{K} \to \mathcal{K}_H$ such that $U_H(v \otimes (X \mathbin{\dot{\otimes}} w)) = (1 \otimes X) \mathbin{\dot{\otimes}} (v \otimes w)$ for all $v \in H$, $X \in \mathcal{P}$ and $w \in H_\theta \otimes K$. Furthermore, the following holds: Let $v \in H$, $(e_i)_{i \in I}$ an orthonormal basis of H, $X \in \mathcal{P}_H$ and $w \in H_\theta \otimes K$. Then $\sum_{i \in I} \|(\omega_{v,e_i} \otimes \iota \otimes \iota)(X) \mathbin{\dot{\otimes}} w\|^2 < \infty$ and*

$$U_H^*(X \mathbin{\dot{\otimes}} (v \otimes w)) = \sum_{i \in I} e_i \otimes \big((\omega_{v,e_i} \otimes \iota \otimes \iota)(X) \mathbin{\dot{\otimes}} w\big) \ .$$

Proof. For $v_1, v_2 \in H$, $X_1, X_2 \in \mathcal{P}$ and $w_1, w_2 \in H_\theta \otimes K$, we have

$$\langle (1 \otimes X_1) \mathbin{\dot{\otimes}} (v_1 \otimes w_1), (1 \otimes X_2) \mathbin{\dot{\otimes}} (v_2 \otimes w_2) \rangle$$
$$= \langle (\iota \otimes \pi_\theta \otimes \iota)((1 \otimes X_2^*)(1 \otimes X_1)) (v_1 \otimes w_1), v_2 \otimes w_2 \rangle$$
$$= \langle (1 \otimes (\pi_\theta \otimes \iota)(X_2^* X_1)) (v_1 \otimes w_1), v_2 \otimes w_2 \rangle$$
$$= \langle v_1, v_2 \rangle \langle (\pi_\theta \otimes \iota)(X_2^* X_1) w_1, w_2 \rangle$$
$$= \langle v_1, v_2 \rangle \langle X_1 \mathbin{\dot{\otimes}} w_1, X_2 \mathbin{\dot{\otimes}} w_2 \rangle = \langle v_1 \otimes (X_1 \mathbin{\dot{\otimes}} w_1), v_2 \otimes (X_2 \mathbin{\dot{\otimes}} w_2) \rangle \ .$$

This implies the existence of an isometric linear map $U_H : H \otimes \mathcal{K} \to \mathcal{K}_H$ such that $U_H(v \otimes (X \mathbin{\dot{\otimes}} w)) = (1 \otimes X) \mathbin{\dot{\otimes}} (v \otimes w)$ for all $v \in H$, $X \in \mathcal{P}$ and $w \in H_\theta \otimes K$.
Choose $X \in \mathcal{P}_H$, $v \in H$ and $w \in H_\theta \otimes K$. Then

$$\sum_{i \in I} \|(\omega_{v,e_i} \otimes \iota \otimes \iota)(X) \mathbin{\dot{\otimes}} w\|^2 = \sum_{i \in I} \langle (\pi_\theta \otimes \iota)((\omega_{v,e_i} \otimes \iota \otimes \iota)(X)^* (\omega_{v,e_i} \otimes \iota \otimes \iota)(X)) w, w \rangle$$
$$\overset{(*)}{=} \langle (\pi_\theta \otimes \iota)((\omega_{v,v} \otimes \iota \otimes \iota)(X^* X)) w, w \rangle = \langle (\iota \otimes \pi_\theta \otimes \iota)(X^* X) (v \otimes w), v \otimes w \rangle < \infty \ ,$$

where we used the normality of $\iota \otimes \pi_\theta \otimes \iota$ and equation (1.1) in (*). Now we get for all $Y \in \mathcal{P}_H$, $v' \in H$ and $w' \in H_\theta \otimes K$ that

$$\Big\langle U_H\Big(\sum_{i \in I} e_i \otimes ((\omega_{v,e_i} \otimes \iota \otimes \iota)(X) \mathbin{\dot{\otimes}} w))\Big), Y \mathbin{\dot{\otimes}} (v' \otimes w') \Big\rangle$$
$$= \sum_{i \in I} \Big\langle U_H\big(e_i \otimes ((\omega_{v,e_i} \otimes \iota \otimes \iota)(X) \mathbin{\dot{\otimes}} w))\big), Y \mathbin{\dot{\otimes}} (v \otimes w) \Big\rangle$$
$$= \sum_{i \in I} \big\langle (1 \otimes (\omega_{v,e_i} \otimes \iota \otimes \iota)(X)) \mathbin{\dot{\otimes}} (e_i \otimes w), Y \mathbin{\dot{\otimes}} (v' \otimes w') \big\rangle$$
$$= \sum_{i \in I} \big\langle (\iota \otimes \pi_\theta \otimes \iota)\big(Y^*(1 \otimes (\omega_{v,e_i} \otimes \iota \otimes \iota)(X))\big) (e_i \otimes w), v' \otimes w' \big\rangle$$
$$= \sum_{i \in I} \big\langle (\pi_\theta \otimes \iota)\big((\omega_{v',e_i} \otimes \iota \otimes \iota)(Y)^* (1 \otimes (\omega_{v,e_i} \otimes \iota \otimes \iota)(X))\big) w, w' \big\rangle$$
$$= \big\langle (\pi_\theta \otimes \iota)((\omega_{v,v'} \otimes \iota \otimes \iota)(Y^* X)) w, w' \big\rangle = \big\langle (\iota \otimes \pi_\theta \otimes \iota)(Y^* X) (v \otimes w), v' \otimes w' \big\rangle$$
$$= \langle X \mathbin{\dot{\otimes}} (v \otimes w), Y \mathbin{\dot{\otimes}} (v' \otimes w') \rangle \ .$$

Hence $U_H\big(\sum_{i\in I} e_i \otimes ((\omega_{v,e_i} \otimes \iota \otimes \iota)(X) \dot{\otimes} w)\big) = X \dot{\otimes} (v \otimes w)$.

From this all we can also conclude that U_H has dense range implying that it is a unitary transformation.

$\square$

Let us now use this proposition to introduce the following notation:

Notation 4.7. *For all $X \in \mathcal{P}_H$, we define the bounded linear operator $X_* : H \otimes H_\theta \otimes K \to H \otimes \mathcal{K}$ such that $X_* v = U_H^*(X \dot{\otimes} v)$ for all $v \in H \otimes H_\theta \otimes K$.*

So we get immediately the following basic facts

1. The mapping $\mathcal{P}_H \to B(H \otimes H_\theta \otimes K, H \otimes \mathcal{K}) : X \to X_*$ is linear and isometric ,
2. $(Y_*)^*(X_*) = (\iota \otimes \pi_\theta \otimes \iota)(Y^*X)$ for all $X, Y \in \mathcal{P}_H$,
3. $H \otimes \mathcal{K} = [\, X_* v \mid X \in \mathcal{P}_H, v \in H \otimes H_\theta \otimes K \,]$.

Notice that the definition of U_H implies that $(1_{B(H)} \otimes X)_* = 1_{B(H)} \otimes X_*$ for all $X \in \mathcal{P}$.

The previous proposition implies for $v \in H$, $(e_i)_{i\in I}$ an orthonormal basis of H, $X \in \mathcal{P}_H$ and $w \in H_\theta \otimes K$ that $\sum_{i\in I} \|(\omega_{v,e_i} \otimes \iota \otimes \iota)(X)_* w\| < \infty$ and

$$X_*(v \otimes w) = \sum_{i\in I} e_i \otimes (\omega_{v,e_i} \otimes \iota \otimes \iota)(X)_* w \ . \tag{4.5}$$

As to be expected, this map $\mathcal{P}_H \to B(H \otimes H_\theta \otimes K, H \otimes \mathcal{K}) : X \to X_*$ also behaves well with respect to the strong topology on bounded sets.

Result 4.8. *Consider a bounded net $(X_i)_{i\in I}$ in $\mathcal{P}_H$ and $X \in \mathcal{P}_H$. Then*

1. *If $(X_i)_{i\in I} \to X$ strongly, then $((X_i)_*)_{i\in I} \to X_*$ strongly.*
2. *If $(X_i)_{i\in I} \to X$ strongly*, then $((X_i)_*)_{i\in I} \to X_*$ strongly*.*

Proof. 1) Since $(X_i)_{i\in I} \to X$ strongly, the net $\big((X_i - X)^*(X_i - X)\big)_{i\in I}$ is a bounded net in $B(H) \otimes Q \otimes B(K)$ that converges to 0 in the weak operator topology. Therefore the normality of $\iota \otimes \pi_\theta \otimes \iota$ implies that the net $\big((\iota \otimes \pi_\theta \otimes \iota)((X_i - X)^*(X_i - X))\big)_{i\in I}$ also converges to 0 in the weak operator topology. Thus $\big(((X_i)_* - X_*)^*((X_i)_* - X_*))\big)_{i\in I}$ converges to 0 in the weak operator topology. It follows that $((X_i)_*)_{i\in I}$ converges strongly to X_*.

2) Choose $Y \in \mathcal{P}_H$. Since $(X_i^*)_{i\in I} \to X^*$ strongly, the net $\big((X_i - X)(X_i - X)^*\big)_{i\in I}$ is a bounded net that converges to 0 in the weak operator topology. Therefore $\big(Y^*(X_i - X)(X_i - X)^*Y\big)_{i\in I}$ is a bounded net in $B(H) \otimes Q \otimes B(K)$ that converges to 0 in the weak operator topology. Thus the normality of $\iota \otimes \pi_\theta \otimes \iota$ implies that the net $\big((\iota \otimes \pi_\theta \otimes \iota)(Y^*(X_i - X)(X_i - X)^*Y)\big)_{i\in I}$ also converges to 0 in the weak operator topology. In other words, the net $\big([((X_i)_* - X_*)^*Y_*]^*[((X_i)_* - X_*)^*Y_*]\big)_{i\in I}$ converges to 0 in the weak operator topology, implying that $\big(((X_i)_*)^*Y_*\big)_{i\in I}$ converges strongly to $(X_*)^*Y_*$. Hence $\big(((X_i)_*)^*(Y_*v)\big)_{i\in I}$ converges to $(X_*)^*(Y_*v)$ for all $v \in H \otimes H_\theta \otimes K$. Because the net $\big(((X_i)_*)^*\big)_{i\in I}$ is bounded and $H \otimes \mathcal{K} = [\, X_* v \mid X \in \mathcal{P}_H, v \in H \otimes H_\theta \otimes K \,]$, we conclude from this all that $\big(((X_i)_*)^*\big)_{i\in I}$ converges strongly to $(X_*)^*$.

$\square$

We will also need the following elementary properties.

Result 4.9. *Consider $X \in \mathcal{P}_H$, then*

1. *$(XY)_* = X_*(\iota \otimes \pi_\theta \otimes \iota)(Y)$ for all $Y \in B(H) \otimes Q \otimes B(K)$,*
2. *$((a \otimes 1 \otimes 1)X)_* = (a \otimes 1)X_*$ for all $a \in B(H)$.*

Proof. 1. Choose $v \in H \otimes H_\theta \otimes K$. We have for all $Z \in \mathcal{P}_H$ and $w \in H \otimes H_\theta \otimes K$ that

$$\langle (XY)_* v, Z_* w \rangle = \langle (\iota \otimes \pi_\theta \otimes \iota)(Z^* XY) v, w \rangle$$
$$= \langle (\iota \otimes \pi_\theta \otimes \iota)(Z^* X)(\iota \otimes \pi_\theta \otimes \iota)(Y) v, w \rangle = \langle X_* (\iota \otimes \pi_\theta \otimes \iota)(Y) v, Z_* w \rangle \; ,$$

from which it follows that $(XY)_* v = X_* (\iota \otimes \pi_\theta \otimes \iota)(Y) v$.

2. Choose $v_1, v_2 \in H$, $w_1, w_2 \in H_\theta \otimes K$ and $Y \in \mathcal{P}$, then

$$\langle \big((a \otimes 1 \otimes 1) X \big)_* (v_1 \otimes w_1), v_2 \otimes Y_* w_1 \rangle = \langle (\iota \otimes \pi_\theta \otimes \iota)((1 \otimes Y)^* (a \otimes 1 \otimes 1) X)(v_1 \otimes w_1), v_2 \otimes w_2 \rangle$$
$$= \langle (\iota \otimes \pi_\theta \otimes \iota)((1 \otimes Y)^* X)(v_1 \otimes w_1), a^* v_2 \otimes w_2 \rangle = \langle X_* (v_1 \otimes w_1), a^* v_2 \otimes Y_* w_2 \rangle$$
$$= \langle (a \otimes 1) X_* (v_1 \otimes w_1), a^* v_2 \otimes Y_* w_2 \rangle \; ,$$

implying that $((a \otimes 1 \otimes 1) X)_* = (a \otimes 1) X_*$ for all $a \in B(H)$. $\qquad \square$

5. THE DEFINITION OF THE INDUCED COREPRESENTATION

In this section we define the induced corepresentation as a partial isometry. In a later section we prove the unitary of this induced corepresentation under an extra (mild?) condition.

Consider $X \in \mathcal{P}$. Using equation (3.2), we see that

$$(\iota \otimes \alpha \otimes \iota)\big((\Delta_M \otimes \iota)(X) \big) = (\Delta_M \otimes \iota \otimes \iota)\big((\alpha \otimes \iota)(X) \big)$$
$$= (\Delta_M \otimes \iota \otimes \iota)(U_{23}^* X_{13}) = U_{34}^* (\Delta_M \otimes \iota)(X)_{124} \; .$$

Consequently, $(\Delta_M \otimes \iota)(X)$ belongs to $\mathcal{P}_{H_M}$.

Proposition 5.1. *There exists a unique isometry $\lambda \in B(H_M \otimes K)$ such that*

$$\lambda(v \otimes X_* w) = (\Delta_M \otimes \iota)(X)_* \, \Upsilon_{12}^* (v \otimes w)$$

for all $v \in H_M$, $X \in \mathcal{P}$ and $w \in H_\theta \otimes K$. Moreover, λ belongs to $M \otimes B(K)$.

Proof. Referring to equation (3.3) we get for all $v_1, v_2 \in H_M$, $w_1, w_2 \in H_\theta \otimes K$ and $X_1, X_2 \in \mathcal{P}$ that

$$\langle (\Delta_M \otimes \iota)(X_1)_* \, \Upsilon_{12}^* (v_1 \otimes w_1), (\Delta_M \otimes \iota)(X_2)_* \, \Upsilon_{12}^* (v_2 \otimes w_2) \rangle$$
$$= \langle (\iota \otimes \pi_\theta \otimes \iota)((\Delta_M \otimes \iota)(X_2)^* (\Delta_M \otimes \iota)(X_1)) \Upsilon_{12}^* (v_1 \otimes w_1), \Upsilon_{12}^* (v_2 \otimes w_2) \rangle$$
$$= \langle \Upsilon_{12} \, (\iota \otimes \pi_\theta \otimes \iota)((\Delta_M \otimes \iota)(X_2^* X_1)) \, \Upsilon_{12}^* (v_1 \otimes w_1), v_2 \otimes w_2 \rangle$$
$$= \langle (1 \otimes (\pi_\theta \otimes \iota)(X_2^* X_1))(v_1 \otimes w_1), v_2 \otimes w_2 \rangle$$
$$= \langle v_1, v_2 \rangle \langle (X_1)_* w_1, (X_2)_* w_2 \rangle = \langle v_1 \otimes (X_1)_* w_1, v_2 \otimes (X_2)_* w_2 \rangle \; .$$

From this chain of equalities the existence of λ follows in the usual way.

Choose $a \in M'$. Take $v \in H_M$, $X \in \mathcal{P}$ and $w \in H_\theta \otimes K$. Then, applying result 4.9 twice and remembering that $(\Delta_M \otimes \iota)(X) \in M \otimes B(H_M) \otimes B(K)$ and $\Upsilon \in M \otimes B(K)$, we get that

$$(\lambda(a \otimes 1))(v \otimes X_* w) = \lambda(av \otimes X^* w) = (\Delta_M \otimes \iota)(X)_* \, \Upsilon_{12}^* (av \otimes w)$$
$$= (\Delta_M \otimes \iota)(X)_* (a \otimes 1 \otimes 1) \, \Upsilon_{12}^* (v \otimes w) = [(\Delta_M \otimes \iota)(X)(a \otimes 1 \otimes 1)]_* \, \Upsilon_{12}^* (v \otimes w)$$
$$= [(a \otimes 1 \otimes 1)(\Delta_M \otimes \iota)(X)]_* \, \Upsilon_{12}^* (v \otimes w) = (a \otimes 1) \, (\Delta_M \otimes \iota)(X)_* \, \Upsilon_{12}^* (v \otimes w)$$
$$= ((a \otimes 1)\lambda)(v \otimes X_* w) \; .$$

Hence $\lambda(a \otimes 1) = (a \otimes 1)\lambda$. We conclude from this that λ belongs to $(M' \otimes 1)' = M \otimes B(K)$. $\qquad \square$

The next proposition establishes that λ^* is a corepresentation of (M, Δ_M) on K.

Proposition 5.2. *We have that* $(\Delta_M \otimes \iota)(\lambda) = \lambda_{23}\,\lambda_{13}$.

Proof. Define $\Delta_M^{(2)} = (\Delta_M \otimes \iota)\Delta_M = (\iota \otimes \Delta_M)\Delta_M$. For all $X \in \mathcal{P}$, we have that

$$(\iota_{B(H \otimes H)} \otimes \alpha \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X)\big) = (\iota \otimes \iota \otimes \alpha \otimes \iota)(\Delta_M \otimes \iota \otimes \iota)(\Delta_M \otimes \iota)(X)$$

$$= (\Delta_M \otimes \iota \otimes \iota \otimes \iota)(\iota \otimes \alpha \otimes \iota)(\Delta_M \otimes \iota)(X) = (\Delta_M \otimes \iota \otimes \iota \otimes \iota)(U_{34}^*(\Delta_M \otimes \iota)(X)_{124})$$

$$= U_{34}^*\,(\Delta_M^{(2)} \otimes \iota)(X)_{124}\ ,$$

implying that $(\Delta_M^{(2)} \otimes \iota)(X)$ belongs to $\mathcal{P}_{H_M \otimes H_M}$.

Let W denote the multiplicative unitary of (M, Δ_M) in the GNS-construction (H_M, ι, Λ_M). We know that $\Delta_M(x) = W^*(1 \otimes x)W$ for all $x \in M$.

Choose $v_1, v_2 \in H_M$, $w \in H_\theta \otimes K$ and $X \in \mathcal{P}$. Fix also an orthonormal basis $(e_i)_{i \in I}$ for H_M.

1) We have that

$$
\begin{aligned}
(\Delta_M \otimes \iota)(\lambda)(v_1 \otimes v_2 \otimes X_* w) &= (W_{12}^* \lambda_{23} W_{12})(v_1 \otimes v_2 \otimes X_* w) \\
&= (W_{12}^* \lambda_{23})(W(v_1 \otimes v_2) \otimes X_* w)\ . \quad (5.6)
\end{aligned}
$$

Choose $u_1, u_2 \in H_M$. Using equation (4.5) in the first and last step of the next chain of equalities, we get for all $p \in H_M$ and $q \in H_\theta \otimes K$ that

$$
\begin{aligned}
&W_{12}^*(u_1 \otimes (\Delta_M \otimes \iota)(X)_*(p \otimes q)) \\
&= \sum_{i \in I} W_{12}^*\big(u_1 \otimes e_i \otimes (\omega_{p,e_i} \otimes \iota \otimes \iota)\big((\Delta_M \otimes \iota)(X)\big)_* q\big) \\
&= \sum_{i \in I}\sum_{j \in I} W_{12}^*\big(e_j \otimes e_i \otimes [(\omega_{u_1,e_j} \otimes \omega_{p,e_i} \otimes \iota \otimes \iota)\big(1 \otimes (\Delta_M \otimes \iota)(X)\big)]_* q\big) \\
&= \sum_{i \in I}\sum_{j \in I} W_{12}^*\big(e_j \otimes e_i \otimes [(\omega_{W^*(u_1 \otimes p),W^*(e_j \otimes e_i)} \otimes \iota \otimes \iota)\big(W_{12}^*(1 \otimes (\Delta_M \otimes \iota)(X))W_{12}\big)]_* q\big) \\
&= \sum_{i \in I}\sum_{j \in I} W^*(e_j \otimes e_i) \otimes [(\omega_{W^*(u_1 \otimes p),W^*(e_j \otimes e_i)} \otimes \iota \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X)\big)]_* q\big) \\
&= (\Delta_M^{(2)} \otimes \iota)(X)_* W_{12}^*(u_1 \otimes p \otimes q)\ .
\end{aligned}
$$

Hence

$$W_{12}^*(u_1 \otimes (\Delta_M \otimes \iota)(X)_*\,\Upsilon_{12}^*(u_2 \otimes w)) = (\Delta_M^{(2)} \otimes \iota)(X)_* W_{12}^*(u_1 \otimes \Upsilon_{12}^*(u_2 \otimes w))\ .$$

This implies that

$$
\begin{aligned}
W_{12}^*\lambda_{23}(u_1 \otimes u_2 \otimes X_* w) &= W_{12}^*(u_1 \otimes (\Delta_M \otimes \iota)(X)_*\,\Upsilon_{12}^*(u_2 \otimes w)) \\
&= (\Delta_M^{(2)} \otimes \iota)(X)_* W_{12}^*(u_1 \otimes \Upsilon_{12}^*(u_2 \otimes w)) = (\Delta_M^{(2)} \otimes \iota)(X)_* W_{12}^* \Upsilon_{23}^*(u_1 \otimes u_2 \otimes w)
\end{aligned}
$$

Combining this with equation (5.6), we find that

$$
\begin{aligned}
(\Delta_M \otimes \iota)(\lambda)(v_1 \otimes v_2 \otimes X_* w) &= (\Delta_M^{(2)} \otimes \iota)(X)_* W_{12}^* \Upsilon_{23}^* W_{12}(v_1 \otimes v_2 \otimes w) \\
&= (\Delta_M^{(2)} \otimes \iota)(X)_*(\Delta_M \otimes \iota)(\Upsilon^*)_{123}(v_1 \otimes v_2 \otimes w) \\
&= (\Delta_M^{(2)} \otimes \iota)(X)_*\,\Upsilon_{23}^*\,\Upsilon_{13}^*(v_1 \otimes v_2 \otimes w)\ , \quad (5.7)
\end{aligned}
$$

where we used equation (3.4) in the last step.

2) Let Σ denote the flip map on $H_M \otimes H_M$. Then we have that

$$
\begin{aligned}
(\lambda_{23}\,\lambda_{13})(v_1 \otimes v_2 \otimes X_* w) &= (\lambda_{23}\Sigma_{12}\lambda_{23}\Sigma_{12})(v_1 \otimes v_2 \otimes X_* w) \\
&= (\lambda_{23}\Sigma_{12}\lambda_{23})(v_2 \otimes v_1 \otimes X_* w) \\
&= (\lambda_{23}\Sigma_{12})\big(v_2 \otimes (\Delta_M \otimes \iota)(X)_*\,\Upsilon_{12}^*(v_1 \otimes w)\big)\ . \quad (5.8)
\end{aligned}
$$

Choose $p \in H_M$, $q \in H_\theta \otimes K$. Then equation (4.5) implies that

$$(\lambda_{23}\Sigma_{12})\big(v_2 \otimes (\Delta_M \otimes \iota)(X)_*(p \otimes q)\big)$$

$$= \sum_{i \in I}(\lambda_{23}\Sigma_{12})(v_2 \otimes e_i \otimes (\omega_{p,e_i} \otimes \iota \otimes \iota)\big((\Delta_M \otimes \iota)(X)\big)_* q)$$

$$= \sum_{i \in I}\lambda_{23}(\, e_i \otimes v_2 \otimes (\omega_{p,e_i} \otimes \iota \otimes \iota)\big((\Delta_M \otimes \iota)(X)\big)_* q\,)$$

$$= \sum_{i \in I} e_i \otimes (\Delta_M \otimes \iota)\big((\omega_{p,e_i} \otimes \iota \otimes \iota)\big((\Delta_M \otimes \iota)(X)\big)\big)_* \Upsilon_{12}^*(v_2 \otimes q)$$

$$= \sum_{i \in I} e_i \otimes (\omega_{p,e_i} \otimes \iota \otimes \iota \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X)\big)_* \Upsilon_{12}^*(v_2 \otimes q) \; . \tag{5.9}$$

Now,

- For all $u \in H_M \otimes H_\theta \otimes K$,

$$\sum_{i \in I}\|(\omega_{p,e_i} \otimes \iota \otimes \iota \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X)\big)_* u\,\|^2$$

$$= \sum_{i \in I}\langle(\iota \otimes \pi_\theta \otimes \iota)\big((\omega_{p,e_i} \otimes \iota \otimes \iota \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X)\big)^*$$

$$(\omega_{p,e_i} \otimes \iota \otimes \iota \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X)\big)\big)\, u, u\rangle$$

$$= \langle(\iota \otimes \pi_\theta \otimes \iota)\big((\omega_{p,p} \otimes \iota \otimes \iota \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X^*X)\big)\big)\, u, u\rangle$$

$$\leq \|p\|^2 \, \|X\|^2 \, \|u\|^2 \; ,$$

implying that the linear map

$$H_M \otimes H_\theta \otimes K \to H_M \otimes H_M \otimes \mathcal{K} : u \mapsto \sum_{i \in I} e_i \otimes (\omega_{p,e_i} \otimes \iota \otimes \iota \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X)\big)_* u$$

is bounded.

- For $u_1 \in H_M$, $u_2 \otimes H_\theta \otimes K$, we get, by applying equation (4.5) twice,

$$\sum_{i \in I} e_i \otimes (\omega_{p,e_i} \otimes \iota \otimes \iota \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X)\big)_*(u_1 \otimes u_2)$$

$$= \sum_{i \in I}\sum_{j \in I} e_i \otimes e_j \otimes (\omega_{p,e_i} \otimes \omega_{u_1,e_j} \otimes \iota \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X)\big)_* u_2$$

$$= (\Delta_M^{(2)} \otimes \iota)(X)_*(p \otimes u_1 \otimes u_2) \; .$$

Combining these fact, we get that

$$\sum_{i \in I} e_i \otimes (\omega_{p,e_i} \otimes \iota \otimes \iota \otimes \iota)\big((\Delta_M^{(2)} \otimes \iota)(X)\big)_* u = (\Delta_M^{(2)} \otimes \iota)(X)_*(p \otimes u)$$

for all $u \in H_M \otimes H_\theta \otimes K$. Using this in combination with the chain of equalities in (5.9), we see that

$$(\lambda_{23}\Sigma_{12})\big(v_2 \otimes (\Delta_M \otimes \iota)(X)_*(p \otimes q)\big) = (\Delta_M^{(2)} \otimes \iota)(X)_* \Upsilon_{23}^*(p \otimes v_2 \otimes q) \; .$$

Hence, by equation (5.8),

$$(\lambda_{23}\lambda_{13})(v_1 \otimes v_2 \otimes X_* w) = (\Delta_M^{(2)} \otimes \iota)(X)_* \Upsilon_{23}^* \Upsilon_{13}^*(v_1 \otimes v_2 \otimes w) \; .$$

Comparing the above equation with equation (5.7), we conclude that

$$(\Delta_M \otimes \iota)(\lambda)(v_1 \otimes v_2 \otimes X_* w) = (\lambda_{23}\,\lambda_{13})(v_1 \otimes v_2 \otimes X_* w) \; .$$

$\square$

As mentioned before, λ is really the adjoint of the induced corepresentation itself:

Notation 5.3. *We define $\rho = \lambda^*$. So ρ is a surjective partial isometry in $M \otimes B(\mathcal{K})$ such that $(\Delta_M \otimes \iota)(\rho) = \rho_{13}\,\rho_{23}$. We call ρ the induced corepresentation associated to the quadruple (M, Δ_M), $(N, \Delta_N), \alpha, U$ with respect to the GNS-construction $(H_\theta, \pi_\theta, \Lambda_\theta)$. We refer to $\mathcal{K}$ as the carrier space of ρ.*

Our definition of the induced corepresentation (and its carrier space) depends on the choice of the n.s.f. weight on Q, but it is no big surprise that all these different induced corepresentations are unitarily equivalent. Let us quickly formalize this statement.

So let η be another n.s.f. weight on Q with GNS-construction $(H_\eta, \pi_\eta, \Lambda_\eta)$. Then u will denote the canonical unitary transformation $u : H_\theta \to H_\eta$ (see e.g. equation (1) in section 3.16 of [17]). Recall that $u\pi_\theta(x)u^* = \pi_\eta(x)$ for all $x \in Q$.

Let ϱ denote the induced corepresentation of the quadruple (M, Δ_M), (N, Δ_N), α, U with respect to the GNS-construction $(H_\eta, \pi_\eta, \Lambda_\eta)$. Define $\mathcal{L}$ to be the carrier space of ϱ

Proposition 5.4. *There exists a unique unitary transformation $\mathcal{U} : \mathcal{K} \to \mathcal{L}$ such that $\mathcal{U}X_* = X_*(u \otimes 1)$ for all $X \in \mathcal{P}$. Let H be any Hilbert space, then we have moreover that $(1 \otimes \mathcal{U})X_* = X_*(1 \otimes u \otimes 1)$ for all $X \in \mathcal{P}_H$.*

Proof. For $X, Y \in \mathcal{P}$ and $v, w \in H_\theta \otimes K$, we have

$$\langle X_*(u \otimes 1)v, Y_*(u \otimes 1)w \rangle = \langle (\pi_\eta \otimes \iota)(Y^*X)(u \otimes 1)v, (u \otimes 1)w \rangle$$
$$= \langle (u^* \otimes 1)(\pi_\eta \otimes \iota)(Y^*X)(u \otimes 1)v, w \rangle = \langle (\pi_\theta \otimes \iota)(Y^*X)v, w \rangle = \langle X_*v, Y_*w \rangle \ .$$

This implies the existence of an isometry $\mathcal{U} : \mathcal{K} \to \mathcal{L}$ such that $\mathcal{U}(X_*v) = X_*(u \otimes 1)v$ for all $X \in \mathcal{P}$ and $v \in H_\theta \otimes K$. It is then immediately clear that $\mathcal{U}$ has dense range and is therefore unitary.

Let H be any Hilbert space and X an element in $\mathcal{P}_H$. Choose $v \in H$, $w \in \otimes H_\theta \otimes K$. Take also an orthonormal basis $(e_i)_{i \in I}$ for H, then

$$\begin{aligned}
(1 \otimes \mathcal{U})X_*(v \otimes w) &= \sum_{i \in I}(1 \otimes \mathcal{U})(e_i \otimes (\omega_{v,e_i} \otimes \iota \otimes \iota)(X)_* w) \\
&= \sum_{i \in I} e_i \otimes (\omega_{v,e_i} \otimes \iota \otimes \iota)(X)_*(u \otimes 1)w \\
&= X_*(v \otimes (u \otimes 1)w) = X_*(1 \otimes u \otimes 1)(v \otimes w) \ ,
\end{aligned}$$

thus $(1 \otimes \mathcal{U})X_* = X_*(1 \otimes u \otimes 1)$. $\qquad\square$

Let Ξ denote the adjoint of the unitary implementation of the left action $\gamma : \pi_\eta(Q) \to M \otimes \pi_\eta(Q)$, defined such that $\gamma(\pi_\eta(a)) = (\iota \otimes \pi_\eta)\Delta_M(x)$ for all $x \in Q$. By proposition 4.1 of [19], we know that $\Xi = (1 \otimes u)\Upsilon(1 \otimes u^*)$. From this we easily infer that

Proposition 5.5. *The corepresentations ρ and ϱ are unitarily equivalent, i.e. $\varrho = (1 \otimes \mathcal{U})\rho(1 \otimes \mathcal{U}^*)$.*

Proof. Choose $X \in \mathcal{P}$, $v \in H_M$, $w \in H_\theta \otimes K$. Then the previous proposition implies that

$$\begin{aligned}
((1 \otimes \mathcal{U})\rho)(v \otimes X_*w) &= (1 \otimes \mathcal{U})(\Delta_M \otimes \iota)(X)_* \Upsilon_{12}^*(v \otimes w) \\
&= (\Delta_M \otimes \iota)(X)_*(1 \otimes u \otimes 1)\Upsilon_{12}^*(v \otimes w) = (\Delta_M \otimes \iota)(X)_* \Xi_{12}^*(1 \otimes u \otimes 1)(v \otimes w) \\
&= \varrho(v \otimes X_*(u \otimes 1)w) = (\varrho(1 \otimes \mathcal{U}))(v \otimes X_*w) \ ,
\end{aligned}$$

and the proposition follows. $\qquad\square$

6. The integrability condition and its consequences for the carrier space $\mathcal{K}$

Recall that the action α is called integrable if the set $\{\, x \in M^+ \mid \alpha(x) \in \mathcal{M}^+_{\iota \otimes \varphi_N} \,\}$ is σ-weakly dense in M^+. In general, an action does not have to be integrable but we will show that our special actions are integrable under a very mild condition (see proposition 6.2). We will show that if our action α is integrable, it is possible to produce an extremely useful dense subset of the carrier space $\mathcal{K}$.

Let us first round up the usual suspects:

1. $\mathcal{M}^+ := \{\, x \in M^+ \mid \alpha(x) \in \mathcal{M}^+_{\iota \otimes \varphi_N} \,\}$, so $\mathcal{M}^+$ is a hereditary cone in M^+,
2. $\mathcal{N} := \{\, x \in M \mid x^* x \in \mathcal{M}^+ \,\}$, so $\mathcal{N}$ is a left ideal in M ,
3. $\mathcal{M} :=$ the linear span of $\mathcal{M}^+$, so $\mathcal{M}$ is a subalgebra of M.

Lemma 6.1.   1. $(\omega \otimes \iota)\Delta_M(x) \in \mathcal{M}^+$ for all $x \in \mathcal{M}^+$ and $\omega \in M_*^+$,
2. $(\omega \otimes \iota)\Delta_M(x) \in \mathcal{M}$ for all $x \in \mathcal{M}$ and $\omega \in M_*$,
3. Let $x \in \mathcal{N}$ and $\omega \in M_*$. Then $(\omega \otimes \iota)\Delta_M(x) \in \mathcal{N}$ and
$$\|(\iota \otimes \Lambda_N)\big(\alpha((\omega \otimes \iota)\Delta_M(x))\big)\| \leq \|\omega\| \, \|(\iota \otimes \Lambda_N)(\alpha(x))\|$$

Proof. 1) Using equation (3.2) we get that
$$\alpha((\omega \otimes \iota)\Delta_M(x)) = (\omega \otimes \iota \otimes \iota)((\iota \otimes \alpha)\Delta_M(x)) = (\omega \otimes \iota \otimes \iota)((\Delta_M \otimes \iota)\alpha(x)) \ .$$
So we get for every $\eta \in M_*^+$ that
$$(\eta \otimes \iota)\big(\alpha((\omega \otimes \iota)\Delta_M(x))\big) = (\,(\omega \otimes \eta)\Delta_M \otimes \iota)(\alpha(x)) \ \in \mathcal{M}^+_{\varphi_N} \ .$$
It follows that $\alpha((\omega \otimes \iota)\Delta_M(x)) \in \mathcal{M}^+_{\iota \otimes \varphi_N}$. Moreover, we have for all $\eta \in M_*^+$ that
$$\begin{aligned}
\eta\big((\iota \otimes \varphi_N)\big(\alpha((\omega \otimes \iota)\Delta_M(x))\big)\big) &= \varphi_N\big((\eta \otimes \iota)\big(\alpha((\omega \otimes \iota)\Delta_M(x))\big)\big) \\
&= (\omega \otimes \eta)\Delta_M((\iota \otimes \varphi_N)\alpha(x)) \ ,
\end{aligned}$$
implying that $(\iota \otimes \varphi_N)\big(\alpha((\omega \otimes \iota)\Delta_M(x))\big) = (\omega \otimes \iota)\Delta_M((\iota \otimes \varphi_N)\alpha(x))$ and hence
$$\|(\iota \otimes \varphi_N)\big(\alpha((\omega \otimes \iota)\Delta_M(x))\big)\| \leq \|\omega\| \, \|(\iota \otimes \varphi)\alpha(x)\| \ .$$

2) This follows immediately from 1.

3) Using the estimate $(\omega \otimes \iota)(\Delta_M(x))^*(\omega \otimes \iota)(\Delta_M(x)) \leq \|\omega\| \, (|\omega| \otimes \iota)\Delta_M(x^*x)$, this is an easy consequence of 1. $\qquad\square$

Now we prove that the integrability of α is easy to check.

Proposition 6.2. *The action α is integrable $\Leftrightarrow$ There exists a non-zero element $x \in M^+$ such that $\alpha(x) \in \mathcal{M}^+_{\iota \otimes \varphi}$.*

Proof. One implication is trivial. We will prove the other one. Therefore suppose that there exists a non-zero element $x \in M^+$ such that $\alpha(x) \in \mathcal{M}^+_{\iota \otimes \varphi}$. Let $\bar{\mathcal{N}}$ denote the σ-weak closure of $\mathcal{N}$ in M. Then $\bar{\mathcal{N}}$ is a σ-weakly closed left ideal in M so there exists a projection P in M such that $\bar{\mathcal{N}} = M P$.
Choose $\omega \in M_*$. By the previous lemma, we know that $(\omega \otimes \iota)\Delta_M(y) \in \mathcal{N}$ for every $y \in \mathcal{N}$. Therefore the normality of $(\omega \otimes \iota)\Delta_M$ implies that $(\omega \otimes \iota)\Delta_M(y) \in \bar{\mathcal{N}}$ for all $y \in \bar{\mathcal{N}}$. In particular, we find that $(\omega \otimes \iota)\Delta_M(P) \in \bar{\mathcal{N}}$ which implies that $(\omega \otimes \iota)\Delta_M(P) = (\omega \otimes \iota)(\Delta_M(P)) \, P$.
From this all we conclude that $\Delta_M(P)(1 \otimes P) = \Delta_M(P)$, thus $\Delta_M(P) \leq 1 \otimes P$. Therefore lemma 6.4 of [11] implies that $P = 0$ or $P = 1$. But the assumption at the beginning of this proof tells us that $P \neq 0$, hence $P = 1$ and $\bar{\mathcal{N}} = M$. $\qquad\square$

For the rest of this paper we assume that our action α is integrable. Thus

1. $\mathcal{M}^+$ is σ-weakly dense in M^+,
2. $\mathcal{M}$ and $\mathcal{N}$ are σ-weakly dense in M.

Using Kaplansky's density theorem, this also implies that all these sets are strongly* dense in M an this in a bounded way, e.g. for every $x \in M$ there exists a net $(x_i)_{i \in I}$ in $\mathcal{M}$ such that $\|x_i\| \leq \|x\|$ for all $i \in I$ and such that $(x_i)_{i \in I}$ converges strongly* to x.

Definition 6.3. *We define the linear map $T_\alpha : \mathcal{M} \to M$ such that $T_\alpha(x) = (\iota \otimes \varphi_N)(\alpha(x))$ for all $x \in \mathcal{M}$.*

Remember that $T_\alpha(x) \geq 0$ for all $x \in \mathcal{M}^+$. It is also clear that for all $x \in \mathcal{M}$ and $a, b \in Q$, the element axb belongs to $\mathcal{M}$ and $T_\alpha(axb) = a \, T_\alpha(x) \, b$.

Proposition 6.4. *The set $T_\alpha(\mathcal{M})$ is a σ-weakly dense two-sided *-ideal of Q.*

Proof. Let $x \in \mathcal{M}$. Then $\alpha(x) \in \mathcal{M}_{\iota \otimes \varphi_N}$. Thus $(\alpha \otimes \iota)\alpha(x) \in \mathcal{M}_{\iota \otimes \iota \otimes \varphi_N}$ and $\alpha((\iota \otimes \varphi_N)\alpha(x)) = (\iota \otimes \iota \otimes \varphi_N)((\alpha \otimes \iota)\alpha(x))$. By equation (3.2), we get that $(\iota \otimes \Delta_N)\alpha(x) \in \mathcal{M}_{\iota \otimes \iota \otimes \varphi_N}$ and

$$(\iota \otimes \iota \otimes \varphi_N)((\iota \otimes \Delta_N)\alpha(x)) = (\iota \otimes \iota \otimes \varphi_N)((\alpha \otimes \iota)\alpha(x)) = \alpha(T_\alpha(x)) \ .$$

By the left invariance of φ_N, we know that $(\iota \otimes \iota \otimes \varphi_N)((\iota \otimes \Delta_N)\alpha(x)) = (\iota \otimes \varphi_N)(\alpha(x)) \otimes 1 = T_\alpha(x) \otimes 1$, thus $\alpha(T_\alpha(x)) = T_\alpha(x) \otimes 1$, so $T_\alpha(x) \in Q$.

So we have proven that T_α is a two-sided ideal in Q. Using the techniques of the proof of proposition 2.5(1) in [6], we arrive at the conclusion that $T_\alpha(\mathcal{M})$ is σ-weakly dense in Q. $\qquad\square$

The techniques used in Chap.1, Sec.3, Cor.5 of Thm.2 of [2], guarantee the following result (see also proposition 5.2(2) of [6]).

Proposition 6.5. *There exists an increasing net $(x_i)_{i \in I}$ in $\mathcal{M}^+$ such that $(T_\alpha(x_i))_{i \in I}$ converges strongly to 1.*

So we see that the map T_α allows us to produce enough elements in Q. We will generalize this construction to produce enough elements in $\mathcal{P}$. We borrowed the basic idea for this procedure from the classical theory of induced group representations but have to use quite different techniques to obtain the relevant results. The starting point is the following basic result.

Proposition 6.6. *Consider $X \in M \otimes B(K)$ such that $U_{23}(\alpha \otimes \iota)(X)$ belongs to $\mathcal{M}_{\iota \otimes \varphi_N \otimes \iota}$. Then $(\iota \otimes \varphi_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))$ belongs to $\mathcal{P}$.*

Proof. Because $U_{23}(\alpha \otimes \iota)(X)$ belongs to $\mathcal{M}_{\iota \otimes \varphi_N \otimes \iota}$, we have that $(\alpha \otimes \iota \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))$ belongs to $\mathcal{M}_{\iota \otimes \iota \otimes \varphi_N \otimes \iota}$ and

$$(\iota \otimes \iota \otimes \varphi_N \otimes \iota)\big((\alpha \otimes \iota \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))\big) = (\alpha \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))\big) \ . \tag{6.10}$$

On the other hand, since $U_{23}(\alpha \otimes \iota)(X) \in \mathcal{M}_{\iota \otimes \varphi_N \otimes \iota}$, the left invariance of φ_N implies that the element $(\iota \otimes \Delta_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))$ belongs to $\mathcal{M}_{\iota \otimes \iota \otimes \varphi_N \otimes \iota}$ and

$$(\iota \otimes \iota \otimes \varphi_N \otimes \iota)\big((\iota \otimes \Delta_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))\big) = (\iota \otimes \varphi_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))_{13} \ .$$

Therefore $U_{24}^*(\iota \otimes \Delta_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))$ belongs to $\mathcal{M}_{\iota \otimes \iota \otimes \varphi_N \otimes \iota}$ and

$$\begin{aligned}
(\iota \otimes \iota \otimes \varphi_N \otimes \iota)&\big(U_{24}^*(\iota \otimes \Delta_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))\big) \\
&= U_{23}^*\,(\iota \otimes \iota \otimes \varphi_N \otimes \iota)\big((\iota \otimes \Delta_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))\big) \\
&= U_{23}^*\,(\iota \otimes \varphi_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))_{13} \ .
\end{aligned} \tag{6.11}$$

Since U is a unitary corepresentation, we get that

$$U_{24}^* (\iota \otimes \Delta_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))$$
$$= U_{24}^* U_{24} U_{34}(\iota \otimes \Delta_N \otimes \iota)((\alpha \otimes \iota)(X)) = U_{34}\, (\iota \otimes \Delta_N \otimes \iota)((\alpha \otimes \iota)(X))\,,$$
$$= U_{34}\, (\alpha \otimes \iota \otimes \iota)((\alpha \otimes \iota)(X)) = (\alpha \otimes \iota \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))\,.$$

Combining this with equations (6.10) and (6.11), we see that $(\iota \otimes \varphi_N \otimes \iota)(U_{23}(\alpha \otimes \iota)(X))$ belongs to $\mathcal{P}$. $\square$

Now the natural question arises how to construct such elements X mentioned in the previous proposition. This will be dealt with in the next 3 results. First we introduce some terminology.

Define the norm continuous one-parameter group σ^* on N_* such that $\sigma_t^*(\omega) = \omega \, \sigma_t^{\psi_N}$ for all $t \in \mathbb{R}$. Let $\omega \in N_*$ and $z \in \mathbb{C}$. Remember that $\omega \in D(\sigma_z^*) \Leftrightarrow$ There exists $\eta \in N_*$ such that $\omega \sigma_z^{\psi_N} \subseteq \eta$. In the latter case, $\sigma_z^*(\omega) = \eta$. Also note that $\omega \in D(\sigma_z^*) \Leftrightarrow \bar{\omega} \in D(\sigma_{\bar{z}}^*)$. If $\omega \in D(\sigma_z^*)$, then $\sigma_{\bar{z}}^*(\bar{\omega}) = \overline{\sigma_z^*(\omega)}$.

We denote the set of elements that are analytic with respect to σ^* by $\mathcal{A}$.

Lemma 6.7. *Let $\omega \in D(\sigma_{\frac{i}{2}}^*)$ and $\eta \in B(K)_*$. Then $(\iota \otimes \eta)([1 \otimes (\omega \otimes \iota)(U^*)]\, U^*)$ belongs to $D(\sigma_{\frac{i}{2}}^{\varphi_N})$ and*

$$\sigma_{\frac{i}{2}}^{\varphi_N}\big((\iota \otimes \eta)([1 \otimes (\omega \otimes \iota)(U^*)]\, U^*)\big) = R\big((\iota \otimes \eta)([1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U)]\, U)\big)\,.$$

Proof. By proposition 6.8 of [11], we get for every $t \in \mathbb{R}$ that

$$\sigma_t^{\varphi_N}\big((\iota \otimes \eta)([1 \otimes (\omega \otimes \iota)(U^*)]\, U^*)\big) = \sigma_t^{\varphi_N}\big((\iota \otimes \eta)((\iota \otimes \omega \otimes \iota)(U_{23}^* U_{13}^*))\big)$$
$$= \sigma_t^{\varphi_N}\big((\iota \otimes \omega \otimes \eta)((\Delta_N \otimes \iota)(U^*))\big) = \sigma_t^{\varphi_N}\big((\iota \otimes \omega)\Delta_N((\iota \otimes \eta)(U^*))\big)$$
$$= (\iota \otimes \omega \sigma_t^{\psi_N})\big((\sigma_t^{\varphi_N} \otimes \sigma_{-t}^{\psi_N})\Delta_N((\iota \otimes \eta)(U^*))\big) = (\iota \otimes \sigma_t^*(\omega))\big(\Delta_N(\tau_t^N((\iota \otimes \eta)(U^*)))\big)\,.$$

Theorem 1.6 (4) of [21] tells us that $(\iota \otimes \eta)(U^*)$ belongs to $D(\tau_{\frac{i}{2}}^N)$ and

$$\tau_{\frac{i}{2}}^N((\iota \otimes \eta)(U^*)) = R((\iota \otimes \eta)(U))\,.$$

Combining these two facts, we see that $(\iota \otimes \eta)([1 \otimes (\omega \otimes \iota)(U^*)]U^*)$ belongs to $D(\sigma_{\frac{i}{2}}^{\varphi_N})$ and

$$\sigma_{\frac{i}{2}}^{\varphi_N}\big((\iota \otimes \eta)([1 \otimes (\omega \otimes \iota)(U^*)]U^*)\big) = (\iota \otimes \sigma_{\frac{i}{2}}^*(\omega)\,)\big(\Delta_N(\tau_{\frac{i}{2}}^N((\iota \otimes \eta)(U^*)))\big)$$
$$= (\iota \otimes \sigma_{\frac{i}{2}}^*(\omega)\,)\Delta_N\big(R((\iota \otimes \eta)(U))\big) = (\iota \otimes \sigma_{\frac{i}{2}}^*(\omega)\,)\big(\chi(R \otimes R)\Delta_N((\iota \otimes \eta)(U))\big)$$
$$= R\big((\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)\Delta_N((\iota \otimes \eta)(U))\,\big) = R\big((\iota \otimes \eta)((\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota \otimes \iota)(U_{13}U_{23}))\,\big)$$
$$= R\big((\iota \otimes \eta)([1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U)]\, U)\big)\,.$$

$\square$

Using the *-operation we get the following variant of this result (needed for later purposes). Let $\omega \in D(\sigma_{-\frac{i}{2}}^*)$ and $\eta \in B(K)_*$. Then $(\iota \otimes \eta)(U\,[1 \otimes (\omega \otimes \iota)(U)])$ belongs to $D(\sigma_{-\frac{i}{2}}^{\varphi_N})$ and

$$\sigma_{-\frac{i}{2}}^{\varphi_N}\big((\iota \otimes \eta)(U\,[1 \otimes (\omega \otimes \iota)(U)])\big) = R\big((\iota \otimes \eta)(U^*\,[1 \otimes (\,\sigma_{-\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U^*)])\big)\,. \tag{6.12}$$

Lemma 6.8. *Define the anti *-automorphism $T : N \to N' : x \mapsto J_N x^* J_N$. Consider $\omega \in D(\sigma_{\frac{i}{2}}^*)$ and $a \in \mathcal{N}_{\varphi_N}$. Then the element $[a \otimes (\omega \otimes \iota)(U^*)]\, U^*$ belongs to $\mathcal{N}_{\varphi_N \otimes \iota}$ and*

$$(\Lambda_N \otimes \iota)\big([a \otimes (\omega \otimes \iota)(U^*)]\, U^*\big) = (1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U))\, (TR \otimes \iota)(U)\, (\Lambda_N(a) \otimes 1)\,.$$

Proof. Take an orthonormal basis $(e_i)_{i \in I}$ for K. Choose $v \in K$. Then the σ-weak lower semi-continuity of φ_N implies that

$$\varphi_N\big((\iota \otimes \omega_{v,v})\big(([a \otimes (\omega \otimes \iota)(U^*)]\, U^*)^* \, [a \otimes (\omega \otimes \iota)(U^*)]\, U^*\big) \big)$$

$$= \varphi_N\big(\sum_{i \in I} (\iota \otimes \omega_{v,e_i})([a \otimes (\omega \otimes \iota)(U^*)]\, U^*)^* \, (\iota \otimes \omega_{v,e_i})([a \otimes (\omega \otimes \iota)(U^*)]\, U^*) \big)$$

$$= \sum_{i \in I} \varphi_N\big((\iota \otimes \omega_{v,e_i})([a \otimes (\omega \otimes \iota)(U^*)]\, U^*)^* \, (\iota \otimes \omega_{v,e_i})([a \otimes (\omega \otimes \iota)(U^*)]\, U^*) \big)$$

$$= \sum_{i \in I} \|\Lambda_N((\iota \otimes \omega_{v,e_i})([a \otimes (\omega \otimes \iota)(U^*)]\, U^*))\|^2$$

$$= \sum_{i \in I} \|\Lambda_N(a\,(\iota \otimes \omega_{v,e_i})([1 \otimes (\omega \otimes \iota)(U^*)]\, U^*))\|^2 \ ,$$

so the previous lemma implies that

$$\varphi_N\big((\iota \otimes \omega_{v,v})\big(([a \otimes (\omega \otimes \iota)(U^*)]\, U^*)^* \, [a \otimes (\omega \otimes \iota)(U^*)]\, U^*\big) \big)$$

$$= \sum_{i \in I} \|T\big(\sigma_{\frac{i}{2}}^{\varphi_N}((\iota \otimes \omega_{v,e_i})([1 \otimes (\omega \otimes \iota)(U^*)]\, U^*))\big) \Lambda_N(a)\|^2 \ ,$$

$$= \sum_{i \in I} \|(TR)\big((\iota \otimes \omega_{v,e_i})([1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U)]\, U)\big) \Lambda_N(a)\|^2$$

$$= \sum_{i \in I} \|(\iota \otimes \omega_{v,e_i})\big((1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U))\, (TR \otimes \iota)(U)\big) \Lambda_N(a)\|^2$$

$$= \sum_{i \in I} \langle\, (\iota \otimes \omega_{v,e_i})\big((1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U))\, (TR \otimes \iota)(U)\big)^*$$

$$(\iota \otimes \omega_{v,e_i})\big((1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U))\, (TR \otimes \iota)(U)\big) \Lambda_N(a), \Lambda_N(a) \rangle$$

$$= \langle\, (\iota \otimes \omega_{v,v})(\, [(1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U))\, (TR \otimes \iota)(U)]^*$$

$$[(1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U))\, (TR \otimes \iota)(U)]\, \Lambda_N(a), \Lambda_N(a) \rangle \ .$$

By the lower semi-continuity of φ_N, this implies that

$$\varphi_N\big((\iota \otimes \omega)\big(([a \otimes (\omega \otimes \iota)(U^*)]\, U^*)^*[a \otimes (\omega \otimes \iota)(U^*)]\, U^*\big) \big)$$

$$= \langle\, (\iota \otimes \omega)(\, [(1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U))\, (TR \otimes \iota)(U)]^*$$

$$[(1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U))(TR \otimes \iota)(U)]\, \Lambda_N(a), \Lambda_N(a) \rangle \ .$$

for all $\omega \in B(K)_*$ from which we conclude that $[a \otimes (\omega \otimes \iota)(U^*)]\, U^*$ belongs to $\mathcal{N}_{\varphi_N \otimes \iota}$. By result 1.2, We have moreover for all $v \in K$ that

$$(\Lambda_N \otimes \iota)([a \otimes (\omega \otimes \iota)(U^*)]\, U^*)\, v$$

$$= \sum_{i \in I} \Lambda_N\big((\iota \otimes \omega_{v,e_i})([a \otimes (\omega \otimes \iota)(U^*)]\, U^*)\big) \otimes e_i$$

$$= \sum_{i \in I} \Lambda_N\big(a\,(\iota \otimes \omega_{v,e_i})([1 \otimes (\omega \otimes \iota)(U^*)]\, U^*)\big) \otimes e_i$$

$$= \sum_{i \in I} T\big(\sigma_{\frac{i}{2}}^{\varphi_N}((\iota \otimes \omega_{v,e_i})([1 \otimes (\omega \otimes \iota)(U^*)]\, U^*))\big) \Lambda_N(a) \otimes e_i$$

$$= \sum_{i \in I} (TR)\big((\iota \otimes \omega_{v,e_i})([1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U)]\, U)\big) \Lambda_N(a) \otimes e_i$$

$$= \sum_{i \in I} (\iota \otimes \omega_{v,e_i})\big((1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U))\, (TR \otimes \iota)(U)\big) \Lambda_N(a) \otimes e_i$$

$$= (1 \otimes (\,\sigma_{\frac{i}{2}}^*(\omega)\, R \otimes \iota)(U))\, (TR \otimes \iota)(U)\, (\Lambda_N(a) \otimes v) \ .$$

$\square$

This lemma implies easily the next one.

Lemma 6.9. *Consider $\omega \in D(\sigma^*_{\frac{i}{2}})$ and $X \in M \otimes N$ such that $X \in \mathcal{N}_{\iota \otimes \varphi_N}$. Then $[X \otimes (\omega \otimes \iota)(U^*)] U^*_{23}$ belongs to $\mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$ and*

$$(\iota \otimes \Lambda_N \otimes \iota)([X \otimes (\omega \otimes \iota)(U^*)] U^*_{23})$$
$$= (1 \otimes 1 \otimes (\sigma^*_{\frac{i}{2}}(\omega) R \otimes \iota)(U)) (TR \otimes \iota)(U)_{23} ((\iota \otimes \Lambda_N)(X) \otimes 1) .$$

Proof. We know that there exists a bounded net $(X_i)_{i \in I}$ in $M \odot \mathcal{N}_{\varphi_N}$ such that $(X_i)_{i \in I}$ converges strongly* to X and $((\iota \otimes \Lambda_N)(X_i))_{i \in I}$ is a bounded net that converges strongly* to $(\iota \otimes \Lambda_N)(X)$. Then $([X_i \otimes (\omega \otimes \iota)(U^*)] U^*_{23})_{i \in I}$ is surely a bounded net that converges strongly* to $[X \otimes (\omega \otimes \iota)(U^*)] U^*_{23}$. It follows easily from the previous lemma that $[X_i \otimes (\omega \otimes \iota)(U^*)] U^*_{23}$ belongs to $\mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$ and

$$(\iota \otimes \Lambda_N \otimes \iota)([X_i \otimes (\omega \otimes \iota)(U^*)] U^*_{23})$$
$$= (1 \otimes 1 \otimes (\sigma^*_{\frac{i}{2}}(\omega) R \otimes \iota)(U)) (TR \otimes \iota)(U)_{23} ((\iota \otimes \Lambda_N)(X_i) \otimes 1) .$$

Therefore the net $((\iota \otimes \Lambda_N \otimes \iota)([X_i \otimes (\omega \otimes \iota)(U^*)] U^*_{23}))_{i \in I}$ is bounded and converges strongly* to

$$(1 \otimes 1 \otimes (\sigma^*_{\frac{i}{2}}(\omega) R \otimes \iota)(U)) (TR \otimes \iota)(U)_{23} ((\iota \otimes \Lambda_N)(X) \otimes 1) .$$

Using the σ-strong*-strong closedness of $\iota \otimes \Lambda_N \otimes \iota$, the lemma follows. $\square$

So we get easily the following two results that will be crucial to us.

Result 6.10. *Consider $x \in \mathcal{N}$ and $\omega \in D(\sigma^*_{-\frac{i}{2}})$. Then $U_{23}(\alpha(x^*) \otimes (\omega \otimes \iota)(U)) \in \mathcal{N}^*_{\iota \otimes \varphi_N \otimes \iota}$.*

Proof. The element $\bar{\omega}$ belongs to $D(\sigma^*_{\frac{i}{2}})$. Hence the previous lemma implies that $[\alpha(x) \otimes (\bar{\omega} \otimes \iota)(U^*)] U^*_{23}$ belongs to $\mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$, implying that $U_{23}[\alpha(x^*) \otimes (\omega \otimes \iota)(U)]$ belongs to $\mathcal{N}^*_{\iota \otimes \varphi_N \otimes \iota}$. $\square$

Remark 6.11. Now proposition 6.6 implies the following results (the second is a special case of the first one).

1. Consider $x \in \mathcal{N}$ and $\omega \in D(\sigma^*_{-\frac{i}{2}})$. Let $X \in M \otimes B(K)$ such that $(\alpha \otimes \iota)(X) \in \mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$. Then $U_{23}(\alpha \otimes \iota)([x^* \otimes (\omega \otimes \iota)(U)] X)$ belongs to $\mathcal{M}_{\iota \otimes \varphi_N \otimes \iota}$ and the element

$$(\iota \otimes \varphi_N \otimes \iota)(U_{23}(\alpha \otimes \iota)([x^* \otimes (\omega \otimes \iota)(U)] X))$$

   belongs to $\mathcal{P}$.

2. Consider $x \in \mathcal{M}$ and $\omega \in D(\sigma^*_{-\frac{i}{2}})$ and $y \in B(K)$ Then the element $U_{23}[\alpha(x) \otimes (\iota \otimes \omega)(U) y]$ belongs to $\mathcal{M}_{\iota \otimes \varphi_N \otimes \iota}$ and the element

$$(\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(x) \otimes (\omega \otimes \iota)(U) y])$$

   belongs to $\mathcal{P}$.

Result 6.12. *Consider $x \in \mathcal{N}$, $X \in \mathcal{P}$ and $\omega \in D(\sigma^*_{\frac{i}{2}})$. Then $(\alpha \otimes \iota)([x \otimes (\omega \otimes \iota)(U^*)] X)$ belongs to $\mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$.*

Proof. By lemma 6.9 we get that $[\alpha(x) \otimes (\omega \otimes \iota)(U^*)] U^*_{23}$ belongs to $\mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$ implying that the element $(\alpha(x) \otimes (\omega \otimes \iota)(U^*)) U^*_{23} X_{13}$ belongs to $\mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$. Because $(\alpha \otimes \iota)(X) = U^*_{23} X_{13}$, the result follows. $\square$

Proposition 6.13. *There exists a directed set I and nets $(a_i)_{i \in I}$ in $\mathcal{N}$, $(\omega_i)_{i \in I}$ in $\mathcal{A}$ such that*

1. $(\iota \otimes \varphi_N \otimes \iota)\big(U_{23}[\alpha(a_i^* a_i) \otimes (\omega_i \otimes \iota)(U)(\omega_i \otimes \iota)(U)^*]U_{23}^*\big) \leq 1$ *for all* $i \in I$.

2. *The net* $\big((\iota \otimes \varphi_N \otimes \iota)\big(U_{23}[\alpha(a_i^* a_i) \otimes (\omega_i \otimes \iota)(U)(\omega_i \otimes \iota)(U)^*]U_{23}^*\big)\big)_{i \in I}$ *converges strongly to 1.*

Proof. Choose $b \in \mathcal{N}$ and $\eta \in \mathcal{A}$. Notice that $\bar{\eta} \in \mathcal{A}$. Using lemma 6.9, we see that

$$(\iota \otimes \varphi_N \otimes \iota)\big(U_{23}[\alpha(b^* b) \otimes (\eta \otimes \iota)(U)(\eta \otimes \iota)(U)^*]U_{23}^*\big)$$
$$= (\iota \otimes \Lambda_N \otimes \iota)\big((\alpha(b) \otimes 1)([1 \otimes (\eta \otimes \iota)(U)^*]U^*)_{23}\big)^*$$
$$(\iota \otimes \Lambda_N \otimes \iota)\big((\alpha(b) \otimes 1)([1 \otimes (\eta \otimes \iota)(U)^*]U^*)_{23}\big)$$
$$= (\iota \otimes \Lambda_N \otimes \iota)\big((\alpha(b) \otimes 1)([1 \otimes (\bar{\eta} \otimes \iota)(U^*)]U^*)_{23}\big)^*$$
$$(\iota \otimes \Lambda_N \otimes \iota)\big((\alpha(b) \otimes 1)([1 \otimes (\bar{\eta} \otimes \iota)(U^*)]U^*)_{23}\big)$$
$$= [(1 \otimes 1 \otimes (\sigma_{\frac{i}{2}}^*(\bar{\eta})\, R \otimes \iota)(U))\,(TR \otimes \iota)(U)_{23}\,((\iota \otimes \Lambda_N)(\alpha(b)) \otimes 1)]^*$$
$$[(1 \otimes 1 \otimes (\sigma_{\frac{i}{2}}^*(\bar{\eta})\, R \otimes \iota)(U))\,(TR \otimes \iota)(U)_{23}\,((\iota \otimes \Lambda_N)(\alpha(b)) \otimes 1)]$$
$$= ((\iota \otimes \Lambda_N)(\alpha(b))^* \otimes 1)\,(TR \otimes \iota)(U^*)_{23}$$
$$(1 \otimes 1 \otimes (\sigma_{\frac{i}{2}}^*(\bar{\eta})\, R \otimes \iota)(U)^* (\sigma_{\frac{i}{2}}^*(\bar{\eta})\, R \otimes \iota)(U))$$
$$(TR \otimes \iota)(U)_{23}((\iota \otimes \Lambda_N)(\alpha(b)) \otimes 1)\,, \tag{6.13}$$

Let us now get hold of some interesting elements:

1. Using proposition 6.5, we get the existence of a net $(x_l)_{l \in L}$ in $\mathcal{M}^+$ such that
   - $(\iota \otimes \varphi_N)(\alpha(x_l)) \leq 1$ for all $l \in L$,
   - The net $\big((\iota \otimes \varphi_N)(\alpha(x_l))\big)_{l \in L}$ converges strongly to 1.

2. Theorem 1.6 of [21] guarantees that the *-algebra $\{(\phi \otimes \iota)(U) \mid \phi \in N_*^\sharp\}$ is a non-degenerate sub*-algebra of $B(K)$ ($N_*^\sharp$ is defined in definition 2.3 of [12]). Hence Kaplansky's density theorem implies the existence of a net $(\phi_j)_{j \in J}$ in $N_*^\sharp$ such that
   - $\|(\phi_j \otimes \iota)(U)\| < 1$ for all $j \in J$,
   - $((\phi_j \otimes \iota)(U))_{j \in J}$ converges strongly* to 1.

Since $\{\sigma_{\frac{i}{2}}^*(\bar{\eta})\, R \mid \eta \in \mathcal{A}\}$ is dense in N_*, this implies easily the existence of a net $(\eta_p)_{p \in P}$ in $\mathcal{A}$ such that
   - $\|(\sigma_{\frac{i}{2}}^*(\bar{\eta}_p)\, R \otimes \iota)(U)\| \leq 1$ for all $p \in P$,
   - The net $\big((\sigma_{\frac{i}{2}}^*(\bar{\eta}_p)\, R \otimes \iota)(U)\big)_{p \in P}$ converges strongly* to 1.

Thus we get that
   - $(\sigma_{\frac{i}{2}}^*(\bar{\eta}_p)\, R \otimes \iota)(U)^* (\sigma_{\frac{i}{2}}^*(\bar{\eta}_p)\, R \otimes \iota)(U) \leq 1$ for all $p \in P$,
   - The net $\big((\sigma_{\frac{i}{2}}^*(\bar{\eta}_p)\, R \otimes \iota)(U)^* (\sigma_{\frac{i}{2}}^*(\bar{\eta}_p)\, R \otimes \iota)(U)\big)_{p \in P}$ converges strongly to 1.

We will use these nets to construct the nets whose existence is claimed in the statement of the proposition. So let $F(H_M \otimes K)$ be the directed set of all finite subsets of $H_M \otimes K$. Set $I = F(H_M \otimes K) \times \mathbb{N}$ and put the product ordering on this set.

Choose $i = (F, n) \in I$. We get first of all the existence of an element $l_i \in L$ such that

$$\|\,[(\iota \otimes \varphi_N)(\alpha(x_{l_i}^* x_{l_i})) \otimes 1)]\,v - v\| \leq \frac{1}{2n}$$

for all $v \in F$.

By the chain of equalities in (6.13), We have for $p \in P$ that

$$(\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(x_{l_i}^* x_{l_i}) \otimes (\eta_p \otimes \iota)(U)(\eta_p \otimes \iota)(U)^*]U_{23}^*)$$
$$= ((\iota \otimes \Lambda_N)(\alpha(x_{l_i}))^* \otimes 1)\,(TR \otimes \iota)(U^*)_{23}$$
$$(1 \otimes 1 \otimes (\sigma_{\frac{i}{2}}^*(\bar{\eta}_p)\, R \otimes \iota)(U)^* (\sigma_{\frac{i}{2}}^*(\bar{\eta}_p)\, R \otimes \iota)(U))$$
$$(TR \otimes \iota)(U)_{23}((\iota \otimes \Lambda_N)(\alpha(x_{l_i})) \otimes 1)\,.$$

This implies that

- We have for $p \in P$ that

$$(\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(x_{l_i}^* x_{l_i}) \otimes (\eta_p \otimes \iota)(U)(\eta_p \otimes \iota)(U)^*]U_{23}^*)$$
$$\leq ((\iota \otimes \Lambda_N)(\alpha(x_{l_i}))^* \otimes 1)(TR \otimes \iota)(U^*)_{23}(TR \otimes \iota)(U)_{23}((\iota \otimes \Lambda_N)(\alpha(x_{l_i})) \otimes 1)$$
$$= (\iota \otimes \varphi_N)(\alpha(x_{l_i}^* x_{l_i})) \otimes 1 \leq 1 .$$

- The net

$$\Big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(x_{l_i}^* x_{l_i}) \otimes (\eta_p \otimes \iota)(U)(\eta_p \otimes \iota)(U)^*]U_{23}^*) \Big)_{p \in p}$$

converges strongly to

$$((\iota \otimes \Lambda_N)(\alpha(x_{l_i}))^* \otimes 1)(TR \otimes \iota)(U^*)_{23}(TR \otimes \iota)(U)_{23}((\iota \otimes \Lambda_N)(\alpha(x_{l_i})) \otimes 1) ,$$

which is equal to $(\iota \otimes \varphi_N)(\alpha(x_{l_i}^* x_{l_i})) \otimes 1$.

From this we infer the existence of an element $p_i \in P$ such that

$$\| (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(x_{l_i}^* x_{l_i}) \otimes (\eta_{p_i} \otimes \iota)(U)(\eta_{p_i} \otimes \iota)(U)^*]U_{23}^*) v$$
$$-((\iota \otimes \varphi_N)(\alpha(x_{l_i}^* x_{l_i})) \otimes 1) v\| \leq \frac{1}{2n}$$

for all $v \in F$.

Now set $a_i = x_{l_i}$ and $\omega_i = \eta_{p_i}$. Then

1. $(\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(a_i^* a_i) \otimes (\omega_i \otimes \iota)(U)(\omega_i \otimes \iota)(U)^*]U_{23}^*) \leq 1$

2. For all $v \in F$,

$$\| (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(a_i^* a_i) \otimes (\omega_i \otimes \iota)(U)(\omega_i \otimes \iota)(U)^*]U_{23}^*) v - v\| \leq \frac{1}{n} .$$

It is also clear that the last inequality implies that the net

$$\Big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(a_i^* a_i) \otimes (\omega_i \otimes \iota)(U)(\omega_i \otimes \iota)(U)^*]U_{23}^*) \Big)_{i \in I}$$

converges strongly to 1. $\qquad\square$

This proposition provides the necessary ammunition to prove the next lemma.

Lemma 6.14. *The carrier space $\mathcal{K}$ is the closed linear span of elements of the form*

$$(\iota \otimes \varphi_N \otimes \iota)\big(U_{23}(\alpha \otimes \iota)([a^* \otimes (\omega \otimes \iota)(U)] X)\big)_* v$$

where $a \in \mathcal{N}$, $\omega \in \mathcal{A}$ and $X \in M \otimes B(K)$ such that $(\alpha \otimes \iota)(X) \in \mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$, $v \in H_\theta \otimes K$.

Proof. Choose $Y \in \mathcal{P}$ and $v \in H_\theta \otimes K$. The previous lemma guarantees the exstence of a directed set I, a net $(a_i)_{i \in I} \in \mathcal{N}$ and a net $(\omega_i)_{i \in I}$ in $\mathcal{A}$ such that

- $(\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(a_i^* a_i) \otimes (\omega_i \otimes \iota)(U)(\omega_i \otimes \iota)(U)^*] U_{23}^*) \leq 1$ for all $i \in I$,
- The net

$$\Big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(a_i^* a_i) \otimes (\omega_i \otimes \iota)(U)(\omega_i \otimes \iota)(U)^*] U_{23}^*) \Big)_{i \in I}$$

converges strongly to 1.

By result 6.12, we have for every $i \in I$ that $(\alpha \otimes \iota)([a_i \otimes (\omega_i \otimes \iota)(U)^*]Y)$ belongs to $\mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$ and

$$(\iota \otimes \varphi_N \otimes \iota)\big(U_{23}(\alpha \otimes \iota)((a_i^* \otimes (\omega_i \otimes \iota)(U))(a_i \otimes (\omega_i \otimes \iota)(U)^*)Y)\big)$$
$$= (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(a_i^* a_i) \otimes (\omega_i \otimes \iota)(U)(\omega_i \otimes \iota)(U)^*]U_{23}^* Y_{13})$$
$$= (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(a_i^* a_i) \otimes (\omega_i \otimes \iota)(U)(\omega_i \otimes \iota)(U)^*]U_{23}^*) Y .$$

From this it follows that

$$\Big((\iota \otimes \varphi_N \otimes \iota)\big(U_{23}(\alpha \otimes \iota)((a_i^* \otimes (\omega_i \otimes \iota)(U))(a_i \otimes (\omega_i \otimes \iota)(U)^*)Y)\big) \Big)_{i \in I}$$

is a bounded net that converges strongly to Y. Referring to result 4.8, we conclude from this that the net

$$\left((\iota \otimes \varphi_N \otimes \iota)\big(U_{23}(\alpha \otimes \iota)((a_i^* \otimes (\omega_i \otimes \iota)(U))(a_i \otimes (\omega_i \otimes \iota)(U)^*)Y)\big)_* v \right)_{i \in I}$$

converges to $Y_* v$. $\qquad\square$

We will combine this result with the next elementary technical lemma on slice weights to get to the penultimate result of this section. It will also be clear that the next lemma holds for any n.s.f. weight on any von Neumann algebra.

Lemma 6.15. *Consider $X \in M \otimes N \otimes B(K)$ such that $X \in \mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$. Then we have for every $\omega \in B(K)_*$ that $(\iota \otimes \iota \otimes \omega)(X)$ belongs to $\mathcal{N}_{\iota \otimes \varphi_N}$. Moreover, the following holds,*
Let $v \in H_M$, $w \in K$ and $(e_i)_{i \in I}$ an orthonormal basis for K, then

$$\sum_{i \in I} \|(\iota \otimes \Lambda_N)((\iota \otimes \iota \otimes \omega_{w,e_i})(X))\, v\| < \infty$$

and

$$(\iota \otimes \Lambda_N \otimes \iota)(X)\,(v \otimes w) = \sum_{i \in I}(\iota \otimes \Lambda_N)((\iota \otimes \iota \otimes \omega_{w,e_i})(X))\, v \otimes e_i \ .$$

Proof. Let us first prove the first statement. Choose $Y \in \mathcal{M}_{\iota \otimes \varphi_N \otimes \iota}^+$ and $\eta \in B(K)_*^+$. Then we have for all $\phi \in M_*^+$ that $(\phi \otimes \iota)\big((\iota \otimes \iota \otimes \eta)(Y)\big) = (\phi \otimes \iota \otimes \eta)(Y)$ which belongs to $\mathcal{M}_{\varphi_N}^+$. This implies that $(\iota \otimes \iota \otimes \eta)(Y)$ belongs to $\mathcal{M}_{\iota \otimes \varphi_N}^+$.
Let $\omega \in B(K)_*$. Using the inequality $(\iota \otimes \iota \otimes \omega)(X)^*(\iota \otimes \iota \otimes \omega)(X) \leq \|\omega\|\,(\iota \otimes \iota \otimes |\omega|)(X^*X)$, the above considerations imply that $(\iota \otimes \iota \otimes \omega)(X)$ belongs to $\mathcal{N}_{\iota \otimes \varphi_N}$.

Now we turn to the second statement. Therefore choose $v \in H_M$, $w \in K$ and $(e_i)_{i \in I}$ an orthonormal basis for K. We have that

$$\sum_{i \in I} \|(\iota \otimes \Lambda_N)((\iota \otimes \iota \otimes \omega_{w,e_i})(X))\, v\|^2 = \sum_{i \in I} \langle (\iota \otimes \varphi_N)\big((\iota \otimes \iota \otimes \omega_{w,e_i})(X)^*(\iota \otimes \iota \otimes \omega_{w,e_i})(X)\big) v, v \rangle \ .$$

Since $\big(\sum_{i \in J}(\iota \otimes \iota \otimes \omega_{w,e_i})(X)^*(\iota \otimes \iota \otimes \omega_{w,e_i})(X)\big)_{J \in F(I)}$ is an increasing net that converges strongly to $(\iota \otimes \iota \otimes \omega_{w,w})(X^*X)$, the above equality and the σ-weak lower semi-continuity of $\iota \otimes \varphi_N$ implies that

$$\sum_{i \in I} \|(\iota \otimes \Lambda_N)((\iota \otimes \iota \otimes \omega_{w,e_i})(X))\, v\|^2 = \langle (\iota \otimes \varphi_N)((\iota \otimes \iota \otimes \omega_{w,w})(X^*X))\, v, v \rangle < \infty \ .$$

Take an orthonormal basis $(f_l)_{l \in L}$ of H_M. Then result 1.2 implies that

$$(\iota \otimes \Lambda_N \otimes \iota)(X)\,(v \otimes w) = \sum_{(i,l) \in I \times L} f_l \otimes \Lambda_N((\omega_{v,f_l} \otimes \iota \otimes \omega_{w,e_i})(X)) \otimes e_i \ ,$$

On the other hand, referring to result 1.2 once again, we get that

$$\begin{aligned}
\sum_{i \in I}(\iota \otimes \Lambda_N)((\iota \otimes \iota \otimes \omega_{w,e_i})(X))\, v \otimes e_i &= \sum_{i \in I}\sum_{l \in L} f_l \otimes \Lambda_N\big((\omega_{v,f_l} \otimes \iota)((\iota \otimes \iota \otimes \omega_{w,e_i})(X))\big) \otimes e_i \\
&= \sum_{i \in I}\sum_{l \in L} f_l \otimes \Lambda_N((\omega_{v,f_l} \otimes \iota \otimes \omega_{w,e_i})(X)) \otimes e_i \ .
\end{aligned}$$

Comparing both expressions, we conclude that

$$(\iota \otimes \Lambda_N \otimes \iota)(X)\,(v \otimes w) = \sum_{i \in I}(\iota \otimes \Lambda_N)((\iota \otimes \iota \otimes \omega_{w,e_i})(X))\, v \otimes e_i \ .$$

$\qquad\square$

Now the proof of our last result is a mere formality.

Proposition 6.16. *The carrier space $\mathcal{K}$ is the closed linear span of elements of the form*

$$(\iota \otimes \varphi_N \otimes \iota)\big(U_{23}[\alpha(x) \otimes (\omega \otimes \iota)(U)]\big)_* v \ ,$$

where $x \in \mathcal{M}$, $\omega \in \mathcal{A}$ and $v \in H_\theta \otimes K$.

Proof. Choose $a \in \mathcal{N}$, $\omega \in \mathcal{A}$, $X \in M \otimes B(K)$ such that $(\alpha \otimes \iota)(X) \in \mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$, $u \in H_\theta$ and $w \in K$. Lemma 6.14 tells us that $\mathcal{K}$ is the closed linear span of elements of the form

$$(\iota \otimes \varphi_N \otimes \iota)\big(U_{23}(\alpha \otimes \iota)([a^* \otimes (\omega \otimes \iota)(U)] X)\big)_*(u \otimes w) \ .$$

Take a basis $(e_i)_{i \in I}$ of K. There exists $\eta \in M_*^+$ such that $\eta(x) = \langle \pi_\theta(x)u, u \rangle$ for all $x \in Q$. Since M is in standard form, we can find an element $q \in H_M$ such that $\eta(x) = \langle xq, q \rangle$ for all $x \in M$.

From the previous lemma, we know that $(\iota \otimes \omega_{w,e_i})(X)$ belongs to $\mathcal{N}$ for all $i \in I$ and that the net

$$\Big(\sum_{i \in J} (\iota \otimes \Lambda_N)\big(\alpha((\iota \otimes \omega_{w,e_i})(X))\big) q \otimes e_i \Big)_{J \in F(I)}$$

converges to $(\iota \otimes \Lambda_N \otimes \iota)((\alpha \otimes \iota)(X))(q \otimes w)$. (*)

Fix $J \in F(I)$ for the moment. Then

$$\big\| (\iota \otimes \varphi_N \otimes \iota)(U_{23}(\alpha \otimes \iota)([a^* \otimes (\omega \otimes \iota)(U)] X))_*(u \otimes w)$$
$$- \sum_{i \in J}(\iota \otimes \varphi_N \otimes \iota)\big(U_{23}(\alpha(a^*(\iota \otimes \omega_{w,e_i})(X)) \otimes (\omega \otimes \iota)(U))\big)_*(u \otimes e_i) \big\|^2$$
$$= \big\| (\iota \otimes \varphi_N \otimes \iota)(U_{23}(\alpha \otimes \iota)([a^* \otimes (\omega \otimes \iota)(U)] X))(q \otimes w)$$
$$- \sum_{i \in J}(\iota \otimes \varphi_N \otimes \iota)\big(U_{23}(\alpha(a^*(\iota \otimes \omega_{w,e_i})(X)) \otimes (\omega \otimes \iota)(U))\big)(q \otimes e_i) \big\|^2$$
$$= \big\|(\iota \otimes \Lambda_N \otimes \iota)([\alpha(a) \otimes (\omega \otimes \iota)(U)^*] U_{23}^*)^*(\iota \otimes \Lambda_N \otimes \iota)(X)(q \otimes w)$$
$$- \sum_{i \in J}(\iota \otimes \Lambda_N \otimes \iota)([\alpha(a) \otimes (\omega \otimes \iota)(U)^*] U_{23}^*)^*((\iota \otimes \Lambda_N)\big(\alpha((\iota \otimes \omega_{w,e_i})(X))\big) q \otimes e_i) \big\|^2 \ .$$

Therefore the convergence in (*) implies that the net

$$\Big(\sum_{i \in J}(\iota \otimes \varphi_N \otimes \iota)\big(U_{23}(\alpha(a^*(\iota \otimes \omega_{w,e_i})(X)) \otimes (\omega \otimes \iota)(U))\big)_*(u \otimes e_i) \Big)_{J \in F(I)}$$

converges to $(\iota \otimes \varphi_N \otimes \iota)(U_{23}(\alpha \otimes \iota)([a^* \otimes (\omega \otimes \iota)(U)] X))_*(u \otimes w)$. Therefore the proposition follows from the considerations in the beginning of the proof. $\qquad \square$

7. Unitarity of the induced corepresentation under the integrability condition

Also in this section we will assume that α is integrable. The aim of this section is to prove the unitarity of the induced corepresentation under this extra assumption.

Define $\mathcal{N}_0 = \langle (\omega \otimes \iota)\Delta_M(a) \mid \omega \in M_*, a \in \mathcal{N} \rangle$. From lemma 6.1 we know that $\mathcal{N}_0 \subseteq \mathcal{N}$ and that

$$\|(\iota \otimes \Lambda_N)\big(\alpha((\omega \otimes \iota)\Delta_M(a))\big)\| \le \|\omega\| \, \|(\iota \otimes \Lambda_N)(\alpha(a))\| \qquad (7.14)$$

for all $a \in \mathcal{N}$ and $\omega \in M_*$ (a fact that we will use several times in this section).

Let us also define a certain closed subspace of $\mathcal{K}$:

Notation 7.1. *We define $\mathcal{K}_0$ as the closed linear span of the elements of the form*

$$(\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y^*x) \otimes (\omega \otimes \iota)(U)])_* v \ ,$$

where $x, y \in \mathcal{N}_0$, $\omega \in \mathcal{A}$ and $v \in H_\theta \otimes K$.

We will ultimately prove that $\mathcal{K} = \mathcal{K}_0$.

Lemma 7.2. *We have that* $\lambda(H_M \otimes \mathcal{K}) \subseteq H_M \otimes \mathcal{K}_0$.

Proof. Choose $x, y \in \mathcal{N}$, $\omega \in \mathcal{A}$, $v \in H_M$ and $w \in H_\theta \otimes K$. Then

$$\lambda\big(v \otimes (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y^*x) \otimes (\omega \otimes \iota)(U)])_* \, w \, \big)$$
$$= (\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y^*x) \otimes (\omega \otimes \iota)(U)])\big)_* \, \Upsilon_{12}^*(v \otimes w) \, . \tag{7.15}$$

Choose an orthonormal basis $(e_i)_{i \in I}$ for H_M. Choose $p \in H_M$ and $q \in H_\theta \otimes K$. By equation (4.5), we know that the net

$$\big(\sum_{i \in J} e_i \otimes (\omega_{p,e_i} \otimes \iota \otimes \iota)\big((\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y^*x) \otimes (\omega \otimes \iota)(U)])\big)\big)_* \, q \big)_{J \in F(I)} \tag{7.16}$$

converges to

$$(\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y^*x) \otimes (\omega \otimes \iota)(U)])\big)_* (p \otimes q) \, .$$

In the next part we will show that each of the sums in this net belongs to $H_M \otimes \mathcal{K}_0$. Therefore fix $j \in I$. Because $U_{23}[\alpha(y^*) \otimes (\omega \otimes \iota)(U)]$ belongs to $\mathcal{N}^*_{\iota \otimes \varphi_N \otimes \iota}$ and $\alpha(x) \otimes 1$ belongs to $\mathcal{N}_{\iota \otimes \varphi_N \otimes \iota}$, we get that $U_{34}[(\Delta_M \otimes \iota)(\alpha(y^*)) \otimes (\omega \otimes \iota)(U)]$ belongs to $\mathcal{N}^*_{\iota \otimes \iota \otimes \varphi_N \otimes \iota}$ and that $(\Delta_M \otimes \iota)(\alpha(x)) \otimes 1$ belongs to $\mathcal{N}_{\iota \otimes \otimes \iota \varphi_N \otimes \iota}$ and

$$(\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y^*x) \otimes (\omega \otimes \iota)(U)])\big)$$
$$= (\iota \otimes \iota \otimes \varphi_N \otimes \iota)\big(U_{34}[(\Delta_M \otimes \iota)(\alpha(y^*)) \otimes (\omega \otimes \iota)(U)] [(\Delta_M \otimes \iota)(\alpha(x)) \otimes 1]\big)$$

This implies that the net

$$\big(\sum_{l \in L} (\iota \otimes \varphi_N \otimes \iota)\big((\omega_{e_l,e_j} \otimes \iota \otimes \iota \otimes \iota)(U_{34}[(\Delta_M \otimes \iota)(\alpha(y^*)) \otimes (\omega \otimes \iota)(U)])$$
$$(\omega_{p,e_l} \otimes \iota \otimes \iota \otimes \iota)((\Delta_M \otimes \iota)(\alpha(x)) \otimes 1)\big) \big)_{L \in F(I)}$$

is a bounded net that converges to

$$(\omega_{p,e_j} \otimes \iota \otimes \iota)\big((\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y^*x) \otimes (\omega \otimes \iota)(U)])\big)\big) \, .$$

Therefore result 4.8 guarantees that the net

$$\big(\sum_{l \in L} (\iota \otimes \varphi_N \otimes \iota)\big((\omega_{e_l,e_j} \otimes \iota \otimes \iota \otimes \iota)(U_{34}[(\Delta_M \otimes \iota)(\alpha(y^*)) \otimes (\omega \otimes \iota)(U)])$$
$$(\omega_{p,e_l} \otimes \iota \otimes \iota \otimes \iota)((\Delta_M \otimes \iota)(\alpha(x)) \otimes 1)\big)_* \, q \, \big)_{L \in F(I)} \tag{7.17}$$

converges to

$$(\omega_{p,e_j} \otimes \iota \otimes \iota)\big((\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y^*x) \otimes (\omega \otimes \iota)(U)])\big)\big)_* \, q \, .$$

But, using equation (3.2), we have for all $l \in L$ that

$$(\iota \otimes \varphi_N \otimes \iota)\big((\omega_{e_l,e_j} \otimes \iota \otimes \iota \otimes \iota)(U_{34}[(\Delta_M \otimes \iota)(\alpha(y^*)) \otimes (\omega \otimes \iota)(U)])$$
$$(\omega_{p,e_l} \otimes \iota \otimes \iota \otimes \iota)((\Delta_M \otimes \iota)(\alpha(x)) \otimes 1)\big)_* \, q$$
$$= (\iota \otimes \varphi_N \otimes \iota)\big((\omega_{e_l,e_j} \otimes \iota \otimes \iota \otimes \iota)(U_{34}[(\iota \otimes \alpha)(\Delta_M(y^*)) \otimes (\omega \otimes \iota)(U)])$$
$$(\omega_{p,e_l} \otimes \iota \otimes \iota \otimes \iota)((\iota \otimes \alpha)(\Delta_M(x)) \otimes 1)\big)_* \, q$$
$$= (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha\big((\omega_{e_j,e_l} \otimes \iota)(\Delta_M(y))^*(\omega_{p,e_l} \otimes \iota)(\Delta_M(x))\big) \otimes (\omega \otimes \iota)(U)])_* \, q,$$

which clearly belongs to $\mathcal{K}_0$. Therefore the convergence in (7.17) implies that

$$(\omega_{p,e_j} \otimes \iota \otimes \iota)\big((\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y^*x) \otimes (\omega \otimes \iota)(U)])\big)\big)_* \, q$$

belongs to $\mathcal{K}_0$. Consequently, by referring to the convergence in (7.16), we see that the element

$$(\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y^*x) \otimes (\omega \otimes \iota)(U)])\big)_*(p \otimes q) \, .$$

The lemma follows now from equation (7.15). □

The next lemma is the last crucial step in the proof of the unitarity of the induced corepresentation.

Lemma 7.3. *We have that* $[\,(a \otimes 1)\lambda v \mid a \in M, v \in H_M \otimes \mathcal{K}_0\,] \subseteq H_M \otimes \mathcal{K}_0$.

Proof. Define the unitary element $V \in B(H_M) \otimes M$ such that $V(\Gamma_M(a) \otimes \Gamma_M(b)) = (\Gamma_M \otimes \Gamma_M)(\Delta_M(a)(1 \otimes b))$ for all $a, b \in \mathcal{N}_{\psi_M}$. Then the following holds

1. $(\omega_{\Gamma_M(a),\Gamma_M(b)} \otimes \iota)(V^*) = (\psi_M \otimes \iota)(\Delta_M(b^*)(a \otimes 1))$ for all $a, b \in \mathcal{N}_{\psi_M}$,
2. $(\iota \otimes \Delta_M)(V) = V_{12}V_{13}$,
3. $\Delta_M(x) = V(x \otimes 1)V^*$ for all $x \in M$.

Call A the norm closure of $\{\,(\omega \otimes \iota)(V) \mid \omega \in B(H_M)_*\,\}$, then A is a σ-weakly dense sub-C*-algebra of M such that V belongs to the multiplier algebra $M(B_0(H_M) \otimes A)$ (for once, the tensor product is here the minimal C*-algebraic tensor product). These last properties follows from the fact that V is a manageable multiplicative unitary in the sense of [21].

First we prove some small technical results

1. Consider a Hilbert space H, $x \in \mathcal{N}$, $v, w \in H_M$ and $a \in B(H_M \otimes H)$. Then the element $(\iota \otimes \alpha)\big((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))\big)$ belongs to $\mathcal{N}_{\iota \otimes \iota \otimes \varphi_N}$ and

$$\|(\iota \otimes \iota \otimes \Lambda_N)\big((\iota \otimes \alpha)\big((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))\big)\big)\| \leq \|v\|\,\|w\|\,\|a\|\,\|(\iota \otimes \Lambda_N)(\alpha(x))\| \ .$$

If $u \in H_M$, we denote by θ_u the element in $B(\mathbb{C}, H_M)$ defined by $\theta_u(c) = c\,u$ for all $c \in \mathbb{C}$. We have that

$$\begin{aligned}
&(\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))^* \, (\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1)) \\
&= [(\theta_{x^*w}^* \otimes 1 \otimes 1)V_{13}^*(a \otimes 1)(\theta_v \otimes 1 \otimes 1)]^* \, [(\theta_{x^*w}^* \otimes 1 \otimes 1)V_{13}^*(a \otimes 1)(\theta_v \otimes 1 \otimes 1)] \\
&= (\theta_v^* \otimes 1 \otimes 1)(a^* \otimes 1)V_{13}(\theta_{x^*w} \otimes 1 \otimes 1)(\theta_{x^*w}^* \otimes 1 \otimes 1)V_{13}^*(a \otimes 1)(\theta_v \otimes 1 \otimes 1) \\
&= (\theta_v^* \otimes 1 \otimes 1)(a^* \otimes 1)V_{13}(x^* \otimes 1 \otimes 1)(\theta_w \theta_w^* \otimes 1 \otimes 1)(x \otimes 1 \otimes 1)V_{13}^*(a \otimes 1)(\theta_v \otimes 1 \otimes 1) \\
&\leq \|w\|^2 \, (\theta_v^* \otimes 1 \otimes 1)(a^* \otimes 1)V_{13}(x^*x \otimes 1 \otimes 1)V_{13}^*(a \otimes 1)(\theta_v \otimes 1 \otimes 1) \\
&= \|w\|^2 \, (\omega_{v,v} \otimes \iota \otimes \iota)((a^* \otimes 1)\Delta_M(x^*x)_{13}(a \otimes 1)) \ .
\end{aligned}$$

Therefore

$$\begin{aligned}
&(\iota \otimes \alpha)\big((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))\big)^* (\iota \otimes \alpha)\big((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))\big) \\
&\quad \leq \|w\|^2 \, (\omega_{v,v} \otimes \iota \otimes \iota \otimes \iota)((a^* \otimes 1 \otimes 1)(\iota \otimes \alpha)(\Delta_M(x^*x))_{134}(a \otimes 1 \otimes 1)) \\
&\quad = \|w\|^2 \, (\omega_{v,v} \otimes \iota \otimes \iota \otimes \iota)((a^* \otimes 1 \otimes 1)(\Delta_M \otimes \iota)(\alpha(x^*x))_{134}(a \otimes 1 \otimes 1)) \ .
\end{aligned}$$

Since $\alpha(x) \in \mathcal{N}_{\iota \otimes \varphi_N}$, this implies immediately that the element

$$(\iota \otimes \alpha)\big((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))\big)^* (\iota \otimes \alpha)\big((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))\big)$$

belongs to $\mathcal{M}_{\iota \otimes \iota \otimes \varphi_N}^+$ and

$$\begin{aligned}
&(\iota \otimes \iota \otimes \varphi_N)\big((\iota \otimes \alpha)\big((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))\big)^* (\iota \otimes \alpha)\big((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))\big)\big) \\
&\quad \leq \|w\|^2 \, (\iota \otimes \iota \otimes \varphi_N)\big((\omega_{v,v} \otimes \iota \otimes \iota \otimes \iota)((a^* \otimes 1 \otimes 1)(\Delta_M \otimes \iota)(\alpha(x^*x))_{134}(a \otimes 1 \otimes 1))\big) \\
&\quad = \|w\|^2 \, (\omega_{v,v} \otimes \iota \otimes \iota)((a^* \otimes 1)\Delta_M((\iota \otimes \varphi_N)\alpha(x^*x))_{13}(a \otimes 1)) \ ,
\end{aligned}$$

implying that the element $(\iota \otimes \alpha)\big((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))\big)$ belongs to $\mathcal{N}_{\iota \otimes \iota \otimes \varphi_N}$ and

$$\begin{aligned}
&\|(\iota \otimes \iota \otimes \Lambda_N)\big((\iota \otimes \alpha)\big((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(a \otimes 1))\big)\big)\|^2 \\
&\quad \leq \|v\|^2\,\|w\|^2\,\|a\|^2\,\|(\iota \otimes \varphi_N)(\alpha(x^*x))\| = \|v\|^2\,\|w\|^2\,\|a\|^2\,\|(\iota \otimes \Lambda_N)(\alpha(x))\|^2 \ .
\end{aligned}$$

2. Consider $x \in \mathcal{N}$ and $v, w \in H_M$. Then $(\omega_{v,x^*w} \otimes \iota)(V^*)$ belongs to $\mathcal{N}$ and

$$\|(\iota \otimes \Lambda_N)\big(\alpha((\omega_{v,x^*w} \otimes \iota)(V^*))\big)\| \leq \|v\| \, \|w\| \, \|(\iota \otimes \Lambda_N)(\alpha(x))\| \ .$$

This is just a special case of the result proven in the previous part.

3. Let $x \in \mathcal{N}$. Then $(\Delta_M \otimes \iota)\alpha(x) \in \mathcal{N}_{\iota \otimes \iota \otimes \varphi_N}$ and

$$\|(\iota \otimes \iota \otimes \Lambda_N)((\Delta_M \otimes \iota)\alpha(x))\| = \|(\iota \otimes \Lambda_N)(\alpha(x))\| \ .$$

We have that $(\Delta_M \otimes \iota)(\alpha(x))^*(\Delta_M \otimes \iota)(\alpha(x)) = (\Delta_M \otimes \iota)\alpha(x^*x)$. Because $\alpha(x^*x)$ belongs to $\mathcal{M}^+_{\iota \otimes \varphi_N}$, this implies that $(\Delta_M \otimes \iota)(\alpha(x))^*(\Delta_M \otimes \iota)(\alpha(x))$ belongs to $\mathcal{M}^+_{\iota \otimes \iota \otimes \varphi_N}$ and

$$\|(\iota \otimes \iota \otimes \varphi_N)\big((\Delta_M \otimes \iota)(\alpha(x))^*(\Delta_M \otimes \iota)(\alpha(x))\,\big)\| = \|\Delta_M((\iota \otimes \varphi_N)\alpha(x^*x))\| = \|(\iota \otimes \varphi_N)\alpha(x^*x)\| \ .$$

Having dealt with these elementary technical issues, we can start with the essential part of the proof. Let $\mathcal{T}$ denote the Tomita-algebra of ψ_N. By properties 2 and 3 above, we get that

$$[\, (\iota \otimes \iota \otimes \Lambda_N)\big((\Delta_M \otimes \iota)\alpha((\omega_{v,x^*w} \otimes \iota)(V^*))\big) \mid x \in \mathcal{N}, v, w \in H_M \,]$$

$$= [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\Delta_M \otimes \iota)\alpha((\omega_{\Gamma_M(bc),x^*\Gamma_M(a)} \otimes \iota)(V^*))\big) \mid x \in \mathcal{N}, a, b \in \mathcal{N}_{\psi_M}, c \in \mathcal{T} \,]$$

$$= [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\Delta_M \otimes \iota)\alpha((\psi_M \otimes \iota)(\Delta_M(a^*x)(bc \otimes 1)))\big) \mid x \in \mathcal{N}, a, b \in \mathcal{N}_{\psi_M}, c \in \mathcal{T} \,]$$

$$= [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\Delta_M \otimes \iota)\alpha((\psi_M \otimes \iota)((\sigma_i^{\psi_N}(c) \otimes 1)\Delta_M(a^*x)(b \otimes 1)))\big) \mid$$
$$\qquad x \in \mathcal{N}, a, b \in \mathcal{N}_{\psi_M}, c \in \mathcal{T} \,]$$

$$\subseteq [\, (\iota \otimes \iota \otimes \Lambda_N)((\Delta_M \otimes \iota)\alpha(y)) \mid y \in \mathcal{N}_0 \,] \ .$$

Therefore,

$$[\, (\iota \otimes \iota \otimes \Lambda_N)((\Delta_M \otimes \iota)\alpha(y))\,(a \otimes 1) \mid y \in \mathcal{N}_0, a \in A \,]$$

$$\supseteq [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\Delta_M \otimes \iota)\alpha((\omega_{v,x^*w} \otimes \iota)(V^*))\big)\,(a \otimes 1) \mid x \in \mathcal{N}, v, w \in H_M, a \in A \,]$$

$$\supseteq [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\Delta_M \otimes \iota)\alpha((\omega_{yv,x^*w} \otimes \iota)(V^*))\big)\,(a \otimes 1) \mid$$
$$\qquad x \in \mathcal{N}, y \in B_0(H_M), v, w \in H_M, a \in A \,]$$

$$\supseteq [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\iota \otimes \alpha)\Delta_M((\omega_{yv,x^*w} \otimes \iota)(V^*))\big)\,(a \otimes 1) \mid$$
$$\qquad x \in \mathcal{N}, y \in B_0(H_M), v, w \in H_M, a \in A \,]$$

$$= [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\iota \otimes \alpha)((\omega_{yv,x^*w} \otimes \iota \otimes \iota)(V_{13}^*V_{12}^*))\big)\,(a \otimes 1) \mid$$
$$\qquad x \in \mathcal{N}, y \in B_0(H_M), v, w \in H_M, a \in A \,]$$

$$= [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\iota \otimes \alpha)((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*[V^*(y \otimes a)]_{12}))\big) \mid$$
$$\qquad x \in \mathcal{N}, y \in B_0(H_M), v, w \in H_M, a \in A \,] \ .$$

Referring to property 1 above, we infer from this that

$$[\, (\iota \otimes \iota \otimes \Lambda_N)((\Delta_M \otimes \iota)\alpha(y))\,(a \otimes 1) \mid y \in \mathcal{N}_0, a \in A \,]$$

$$\supseteq [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\iota \otimes \alpha)((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*(V^*z)_{12}))\big) \mid$$
$$\qquad x \in \mathcal{N}, z \in B_0(H_M) \otimes A, v, w \in H_M \,]$$

$$= [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\iota \otimes \alpha)((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*\, z_{12}))\big) \mid$$
$$\qquad x \in \mathcal{N}, z \in B_0(H_M) \otimes A, v, w \in H_M \,]$$

$$\supseteq [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\iota \otimes \alpha)((\omega_{v,x^*w} \otimes \iota \otimes \iota)(V_{13}^*\, (y \otimes a)_{12}))\big) \mid$$
$$\qquad x \in \mathcal{N}, y \in B_0(H_M), v, w \in H_M, a \in A \,]$$

$$= [\, (\iota \otimes \iota \otimes \Lambda_N)\big((\iota \otimes \alpha)(a \otimes (\omega_{yv,x^*w} \otimes \iota)(V^*))\big) \mid$$
$$\qquad x \in \mathcal{N}, y \in B_0(H_M), v, w \in H_M, a \in A \,]$$

$$= [\, a \otimes (\iota \otimes \Lambda_N)\big(\alpha((\omega_{yv,x^*w} \otimes \iota)(V^*))\big) \mid x \in \mathcal{N}, y \in B_0(H_M), v, w \in H_M, a \in A \,] \ .$$

Hence, invoking property 2 above, we conclude that

$$[\,(\iota \otimes \iota \otimes \Lambda_N)((\Delta_M \otimes \iota)\alpha(y))\,(a \otimes 1) \mid y \in \mathcal{N}_0, a \in A\,]$$
$$\supseteq [\,a \otimes (\iota \otimes \Lambda_N)\big(\alpha((\omega_{v,x^*w} \otimes \iota)(V^*))\big) \mid x \in \mathcal{N}, v, w \in H_M, a \in A\,]$$
$$\supseteq [\,a \otimes (\iota \otimes \Lambda_N)\big(\alpha((\omega_{\Gamma_M(cd), x^*\Gamma_M(b)} \otimes \iota)(V^*))\big) \mid x \in \mathcal{N}, b, c \in \mathcal{N}_{\psi_M}, d \in \mathcal{T}, a \in A\,]$$
$$= [\,a \otimes (\iota \otimes \Lambda_N)\big(\alpha((\psi_M \otimes \iota)(\Delta_M(b^*x)(cd \otimes 1)))\big) \mid x \in \mathcal{N}, b, c \in \mathcal{N}_{\psi_M}, d \in \mathcal{T}, a \in A\,]$$
$$= [\,a \otimes (\iota \otimes \Lambda_N)\big(\alpha((\psi_M \otimes \iota)((\sigma_i^{\psi_M}(d) \otimes 1)\Delta_M(b^*x)(c \otimes 1)))\big) \mid$$
$$x \in \mathcal{N}, b, c \in \mathcal{N}_{\psi_M}, d \in \mathcal{T}, a \in A\,],$$

so that inequality (7.14) us lets conclude that

$$[\,(\iota \otimes \iota \otimes \Lambda_N)((\Delta_M \otimes \iota)\alpha(y))\,(a \otimes 1) \mid y \in \mathcal{N}_0, a \in A\,]$$
$$\supseteq [\,a \otimes (\iota \otimes \Lambda_N)\big(\alpha((\omega_{p,q} \otimes \iota)\Delta_M(b^*x))\big) \mid x \in \mathcal{N}, b \in \mathcal{N}_{\psi_M}, p, q \in H_M, a \in A\,] . \tag{7.18}$$

Now,

$$[\,(a^* \otimes 1)\lambda v \mid a \in A, v \in H_M \otimes \mathcal{K}_0\,]$$
$$= [\,(a^* \otimes 1)\lambda\big(u \otimes (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(x_2^*x_1) \otimes (\eta \otimes \iota)(U)])_* w\big) \mid$$
$$a \in A, x_1, x_2 \in \mathcal{N}_0, u \in H_M, w \in H_\theta \otimes K, \eta \in \mathcal{A}\,]$$
$$= [\,(a^* \otimes 1)(\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(x_2^*x_1) \otimes (\eta \otimes \iota)(U)])\big)_* \Upsilon_{12}^*(u \otimes w) \mid$$
$$a \in A, x_1, x_2 \in \mathcal{N}_0, u \in H_M, w \in H_\theta \otimes K, \eta \in \mathcal{A}\,]$$
$$= [\,(a^* \otimes 1)(\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(x_2^*x_1) \otimes (\eta \otimes \iota)(U)])\big)_* (u \otimes w) \mid$$
$$a \in A, x_1, x_2 \in \mathcal{N}_0, u \in H_M, w \in H_\theta \otimes K, \eta \in \mathcal{A}\,]$$
$$= [\,(a_2^* \otimes 1)(\Delta_M \otimes \iota)\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(x_2^*x_1) \otimes (\eta \otimes \iota)(U)])\big)_* (a_1 u \otimes w) \mid$$
$$a_1, a_2 \in A, x_1, x_2 \in \mathcal{N}_0, u \in H_M, w \in H_\theta \otimes K, \eta \in \mathcal{A}\,]$$
$$= [\,\big((a_2^* \otimes 1 \otimes 1)(\iota \otimes \iota \otimes \varphi_N \otimes \iota)\big(U_{34}[(\Delta_M \otimes \iota)(\alpha(x_2^*x_1)) \otimes (\eta \otimes \iota)(U)]\big)$$
$$(a_1 \otimes 1 \otimes 1)\big)_* (u \otimes w) \mid a_1, a_2 \in A, x_1, x_2 \in \mathcal{N}_0, u \in H_M, w \in H_\theta \otimes K, \eta \in \mathcal{A}\,]$$

For $\eta \in \mathcal{A}$, we define the element $Z_\eta \in B(H_N \otimes K)$ as

$$Z_\eta = (TR \otimes \iota)(U^*)\,(1 \otimes (\,\sigma_{\frac{i}{2}}^*(\bar{\eta})\,R \otimes \iota)(U)^*) .$$

By lemma 6.9, we have for $a_1, a_2 \in A$, $x_1, x_2 \in \mathcal{N}_0$ and $\eta \in \mathcal{A}$ that

$$(a_2^* \otimes 1 \otimes 1)(\iota \otimes \iota \otimes \varphi_N \otimes \iota)\big(U_{34}[(\Delta_M \otimes \iota)(\alpha(x_2^*x_1)) \otimes (\eta \otimes \iota)(U)]\big)(a_1 \otimes 1 \otimes 1)$$
$$= (\,[(\iota \otimes \iota \otimes \Lambda_N)((\Delta_M \otimes \iota)\alpha(x_2))\,(a_2 \otimes 1)]^* \otimes 1)$$
$$(Z_\eta)_{34}\,(\,[(\iota \otimes \iota \otimes \Lambda_N)((\Delta_M \otimes \iota)\alpha(x_1))\,(a_1 \otimes 1)] \otimes 1)$$

Similarly, using lemma 6.9, we have for $x_1, x_2 \in \mathcal{N}_0$ and $\eta \in \mathcal{A}$ that

$$(\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(x_2^*x_1) \otimes (\eta \otimes \iota)(U)])$$
$$= ((\iota \otimes \Lambda_N)(\alpha(x_2))^* \otimes 1)\,(Z_\eta)_{23}\,((\iota \otimes \Lambda_N)(\alpha(x_1)) \otimes 1) . \tag{7.19}$$

Therefore inclusion (7.18) implies that

$$[\,(a^* \otimes 1)\lambda v \mid a \in A, v \in H_M \otimes \mathcal{K}_0\,]$$
$$\supseteq [\,a_2^*a_1 u \otimes (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha\big((\omega_{p_2,q_2} \otimes \iota)(\Delta_M(b_2^*x_2))^*$$
$$(\omega_{p_1,q_1} \otimes \iota)(\Delta_M(b_1^*x_1))\big) \otimes (\eta \otimes \iota)(U)])_* w \mid a_1, a_2 \in A, x_1, x_2 \in \mathcal{N},$$
$$b_1, b_2 \in \mathcal{N}_{\psi_M}, u \in H_M, w \in H_\theta \otimes K, \eta \in \mathcal{A}, p_1, p_2, q_1, q_2 \in H_M\,] \tag{7.20}$$

Take a bounded net $(f_i)_{i \in I}$ in $\mathcal{N}_{\psi_M}$ such that $(f_i)_{i \in I}$ converges strongly* to 1.

Let $p, q \in H_M$ and $x \in \mathcal{N}$. Then

$$(\iota \otimes \Lambda_N)\big(\alpha((\omega_{p,q} \otimes \iota)\Delta_M(f_i^* x))\big) = (\theta_q^* \otimes 1 \otimes 1)(\iota \otimes \iota \otimes \Lambda_N)((\iota \otimes \alpha)\Delta_M(f_i^* x))(\theta_p \otimes 1 \otimes 1)$$
$$= (\theta_q^* \otimes 1 \otimes 1)(\iota \otimes \alpha)(\Delta_M(f_i^*))(\iota \otimes \iota \otimes \Lambda_N)((\iota \otimes \alpha)\Delta_M(x))(\theta_p \otimes 1 \otimes 1) \; ,$$

so that the normality of $(\iota \otimes \alpha)\Delta_M$ implies that $\big((\iota \otimes \Lambda_N)\big(\alpha((\omega_{p,q} \otimes \iota)\Delta_M(f_i^* x))\big) \big)_{i \in I}$ is a bounded net that converges strongly* to $(\iota \otimes \Lambda_N)\big(\alpha((\omega_{p,q} \otimes \iota)\Delta_M(x))\big)$

Using this fact and formula (7.19), we get for all $x_1, x_2 \in \mathcal{N}$ and $p_1, p_2, q_1, q_2 \in H_M$ that the net

$$\big((\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha\big((\omega_{p_2,q_2} \otimes \iota)(\Delta_M(f_i^* x_2))^*(\omega_{p_1,q_1} \otimes \iota)(\Delta_M(f_i^* x_1))\big) \otimes (\eta \otimes \iota)(U)]) \big)_{i \in I}$$

is bounded and converges strongly* to

$$(\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha\big((\omega_{p_2,q_2} \otimes \iota)(\Delta_M(x_2))^*(\omega_{p_1,q_1} \otimes \iota)(\Delta_M(x_1))\big) \otimes (\eta \otimes \iota)(U)]) \; .$$

Therefore result 4.8 and inclusion (7.20) above imply that

$$[\, (a^* \otimes 1)\lambda v \mid a \in A, v \in H_M \otimes \mathcal{K}_0 \,]$$
$$\supseteq [\, u \otimes (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha\big((\omega_{p_2,q_2} \otimes \iota)(\Delta_M(x_2))^*$$
$$(\omega_{p_1,q_1} \otimes \iota)(\Delta_M(x_1))\big) \otimes (\eta \otimes \iota)(U)])_* \, w \mid x_1, x_2 \in \mathcal{N}, u \in H_M,$$
$$w \in H_\theta \otimes K, \eta \in \mathcal{A}, p_1, p_2, q_1, q_2 \in H_M \,]$$

Because M is supposed to be in standard form, this becomes

$$[\, (a^* \otimes 1)\lambda v \mid a \in A, v \in H_M \otimes \mathcal{K}_0 \,]$$
$$\supseteq [\, u \otimes (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha\big((\omega_2 \otimes \iota)(\Delta_M(x_2))^*$$
$$(\omega_1 \otimes \iota)(\Delta_M(x_1))\big) \otimes (\eta \otimes \iota)(U)])_* \, w \mid x_1, x_2 \in \mathcal{N}, u \in H_M,$$
$$w \in H_\theta \otimes K, \eta \in \mathcal{A}, \omega_1, \omega_2 \in M_* \,]$$
$$= [\, u \otimes (\iota \otimes \varphi_N \otimes \iota)(U_{23}[\alpha(y_2^* y_1) \otimes (\eta \otimes \iota)(U)])_* \, w \mid$$
$$u \in H_M, w \in H_\theta \otimes K, y_1, y_2 \in \mathcal{N}_0, \eta \in \mathcal{A} \,]$$
$$= H_M \otimes \mathcal{K}_0 \; .$$

$\square$

Lemma 7.4. *We have that $\lambda(H_M \otimes \mathcal{K}_0) = H_M \otimes \mathcal{K}_0$.*

Proof. Define p to be the projection on $\mathcal{K}_0$. Also define $\phi = \lambda(1 \otimes p)$, then $\phi^* \phi = (1 \otimes p)\lambda^* \lambda(1 \otimes p) = 1 \otimes p$ implying that ϕ is a partial isometry in $M \otimes B(\mathcal{K})$. Define $P = \phi\phi^* = \lambda(1 \otimes p)\lambda^* \in M \otimes B(\mathcal{K})$ to be the final projection of ϕ. Thus, since $(\Delta_M \otimes \iota)(\lambda) = \lambda_{23}\lambda_{13}$, we get that

$$(\Delta_M \otimes \iota)(P) = (\Delta_M \otimes \iota)(\lambda(1 \otimes p)\lambda^*) = \lambda_{23}\lambda_{13}(1 \otimes 1 \otimes p)\lambda_{13}^* \lambda_{23}^* = \lambda_{23}P_{13}\lambda_{23}^* \; .$$

Because $\lambda(H_M \otimes \mathcal{K}_0) \subseteq H_M \otimes \mathcal{K}_0$, we have that $\lambda(1 \otimes p) = (1 \otimes p)\lambda(1 \otimes p)$, implying that $P(1 \otimes p) = \lambda(1 \otimes p)\lambda^*(1 \otimes p) = \lambda(1 \otimes p)\lambda^* = P$. Hence

$$(\Delta_M \otimes \iota)(P)\, P_{23} = \lambda_{23}P_{13}\lambda_{23}^* \lambda_{23}(1 \otimes 1 \otimes p)\lambda_{23}^*$$
$$= \lambda_{23}P_{13}(1 \otimes 1 \otimes p)\lambda_{23}^* = \lambda_{23}P_{13}\lambda_{23}^* = (\Delta_M \otimes \iota)(P) \; .$$

Arguing as in the proof of lemma 6.4 of [11], we conclude from this that $(\Delta_M \otimes \iota)(P) = P_{23}$. So we get for every $\omega \in B(\mathcal{K})_*$ that $\Delta_M((\iota \otimes \omega)(P)) = 1 \otimes (\iota \otimes \omega)(P)$ and thus $(\iota \otimes \omega)(P) \in \mathbb{C}1$ by result 5.13 of [11]. This implies that $P \in (B(H_M) \otimes 1)' = 1 \otimes B(\mathcal{K})$. Therefore there exists a closed subspace $\mathcal{K}_1$ of $\mathcal{K}$ such that P is the orthogonal projection on $H_M \otimes \mathcal{K}_1$, thus $\lambda(H_M \otimes \mathcal{K}_0) = \phi(H_M \otimes \mathcal{K}_0) = H_M \otimes \mathcal{K}_1$. Hence the previous proposition implies that

$$H_M \otimes \mathcal{K}_0 = [\, (a \otimes 1)\lambda v \mid a \in M, v \in H_M \otimes \mathcal{K}_0 \,] = [\, (a \otimes 1)w \mid a \in M, w \in H_M \otimes \mathcal{K}_1 \,] = H_M \otimes \mathcal{K}_1,$$

thus $\mathcal{K}_1 = \mathcal{K}_0$ and the lemma follows. $\qquad\square$

Now we only have to tie up the loose ends to arrive at our final conclusion (we still work under the integrability condition of the beginning of the section!):

Proposition 7.5. *The induced corepresentation ρ associated to the quadruple (M, Δ_M), (N, Δ_N), α, U with respect to $(H_\theta, \pi_\theta, \Lambda_\theta)$ is a unitary element in $M \otimes B(\mathcal{K})$.*

Proof. By lemma 7.2 and the previous lemma, we know that

$$\lambda(H_M \otimes \mathcal{K}) \subseteq H_M \otimes \mathcal{K}_0 \subseteq \lambda(H_M \otimes \mathcal{K}_0) \ .$$

Since λ is an isometry, this implies that $H_M \otimes \mathcal{K} \subseteq H_M \otimes \mathcal{K}_0$ and hence $\mathcal{K} = \mathcal{K}_0$. Using the previous lemma once more, we get that $\lambda(H_M \otimes \mathcal{K}) = H_M \otimes \mathcal{K}$ so that λ is unitary. Because $\rho = \lambda^*$, the proposition follows. $\qquad\square$

It is also worthwhile remembering that $\mathcal{K} = \mathcal{K}_0$.

8. A correspondence between weights on Q and certain weights on M

Also in this section we will assume that α is integrable. Extend the function $\mathcal{M}^+ \to Q^+ : x \mapsto T_\alpha(x)$ to a function $\mathcal{T}_\alpha : M^+ \to M^+_{\text{ext}}$ such that $\mathcal{T}_\alpha(x) = (\iota \otimes \varphi_N)(\alpha(x))$ for all $x \in M^+$. Here we consider the map $\iota \otimes \varphi_N : (M \otimes N)^+ \to M^+_{\text{ext}}$ as an operator valued weight. The positive extended part Q^+_{ext} is naturally embedded in M^+_{ext} (see proposition 1.9 of [6]). It is proven in proposition 1.3 of [19] that, under this embedding, $\mathcal{T}_\alpha(M^+) \subseteq Q^+_{\text{ext}}$ and that $\mathcal{T}_\alpha : M^+ \to Q^+_{\text{ext}}$ is a semi-finite operator valued weight. We use this operator valued weight $\mathcal{T}_\alpha$ to pull weights down from Q to M (we will use proposition 2.3 of [6] for this):

Definition 8.1. *Consider a n.s.f. weight η on Q. Then we define the n.s.f. weight $\tilde{\eta}$ on M such that $\tilde{\eta}(x) = \eta(\mathcal{T}_\alpha(x))$ for all $x \in M^+$.*

For any normal weight η on Q and any element $x \in \hat{Q}_+$ the element $\eta(x) \in [0, \infty]$ is defined in such a way that the following holds. Let $(\eta_i)_{i \in I}$ be a family of elements in Q^+_* such that $\eta(y) = \sum_{i \in I} \eta_i(y)$ for all $y \in Q^+$ (such a family always exist). Then $\eta(x) = \sum_{i \in I} \eta_i(x)$.

Such pulled down weights satisfy a natural invariance condition with respect to α (see proposition 2.8 of [4]).

Proposition 8.2. *Consider a n.s.f. weight η on Q, $a \in \mathcal{M}_{\tilde{\eta}}$ and $v, w \in D(\delta_N^{\frac{1}{2}})$. Then $(\iota \otimes \omega_{v,w})\alpha(a)$ belongs to $\mathcal{M}_{\tilde{\eta}}$ and*

$$\tilde{\eta}\big((\iota \otimes \omega_{v,w})\alpha(a)\big) = \langle \delta_N^{\frac{1}{2}} v, \delta_N^{\frac{1}{2}} w \rangle \, \tilde{\eta}(a) \ .$$

A similar result holds for ψ_M:

Proposition 8.3. *Consider $a \in \mathcal{M}_{\psi_M}$ and $\omega \in N_*$. Then $(\iota \otimes \omega)\alpha(a) \in \mathcal{M}^+_{\psi_M}$ and*

$$\psi_M((\iota \otimes \omega)\alpha(a)) = \psi_M(a) \, \omega(1) \ .$$

Proof. We may assume that $a \geq 0$ and $\omega \geq 0$. We have for $\omega' \in M_*^+$ that

$$(\iota \otimes \omega')\Delta_M((\iota \otimes \omega)\alpha(a)) = (\iota \otimes \omega' \otimes \omega)((\Delta_M \otimes \iota)\alpha(a)) = (\iota \otimes (\omega' \otimes \omega)\alpha)\Delta_M(a) \ .$$

Therefore the right invariance of ψ_M implies that $(\iota \otimes \omega')\Delta_M((\iota \otimes \omega)\alpha(a)) \in \mathcal{M}_{\psi_M}^+$.
Using proposition 5.14 of [11], we conclude from this that $(\iota \otimes \omega)\alpha(a)$ belongs to $\mathcal{M}_{\psi_M}^+$. By right invariance of ψ_M we have moreover for all $\omega' \in M_*^+$ that

$$
\begin{aligned}
\psi_M((\iota \otimes \omega)\alpha(a))\,\omega'(1) &= \psi_M\big((\iota \otimes \omega')\Delta_M((\iota \otimes \omega)\alpha(a))\big) = \psi_M\big((\iota \otimes (\omega' \otimes \omega)\alpha)\Delta_M(a)\big) \\
&= \psi_M(a)\,(\omega' \otimes \omega)(\alpha(1)) = \psi_M(a)\,\omega(1)\,\omega'(1) \ ,
\end{aligned}
$$

implying that $\psi_M((\iota \otimes \omega)\alpha(a)) = \psi_M(a)\,\omega(1)$. $\qquad\square$

Consider a n.f.s. weight ϕ on M and a positive self-adjoint operator γ affiliated to N such that $\Delta_N(\gamma) = \gamma \otimes \gamma$ and for all $a \in \mathcal{M}_\phi$ and all $v, w \in D(\gamma^{\frac{1}{2}})$, we have that the element $(\iota \otimes \omega_{v,w})\alpha(a)$ belongs to $\mathcal{M}_\phi$ and

$$\phi\big((\iota \otimes \omega_{v,w})\alpha(a)\big) = \phi(a)\,\langle \gamma^{\frac{1}{2}}v, \gamma^{\frac{1}{2}}w \rangle \ .$$

Choose also a GNS-construction $(H_\phi, \pi_\phi, \Lambda_\phi)$ for ϕ. In the next paragraphs we will state some results without proof because we believe the reader has acquired the necessary skills and techniques by now to check these results him or herself.

First of all, we have for $a \in \mathcal{N}_\phi$, $v \in D(\gamma^{\frac{1}{2}})$ and $w \in H_N$ that $(\iota \otimes \omega_{v,w})\alpha(a) \in \mathcal{N}_\phi$ and

$$\|\Lambda_\phi((\iota \otimes \omega_{v,w})\alpha(a))\| \leq \|\Lambda_\phi(a)\|\,\|\gamma^{\frac{1}{2}}v\|\,\|w\| \ .$$

Let $(e_i)_{i \in I}$ be an orthonormal basis for H_N. Then we have for $a \in \mathcal{N}_\phi$ and $v \in D(\gamma^{\frac{1}{2}})$ that

$$\sum_{i \in I} \|\Lambda_\phi((\iota \otimes \omega_{v,e_i})\alpha(a))\|^2 = \|\Lambda_\phi(a)\|^2\,\|\gamma^{\frac{1}{2}}v\|^2 < \infty \ .$$

Therefore we can define an isometry $V_\phi \in B(H_\phi \otimes H_N)$ such that

$$V_\phi\,(\Lambda_\phi(a) \otimes v) = \sum_{i \in I} \Lambda_\phi\big((\iota \otimes \omega_{\gamma^{-\frac{1}{2}}v,e_i})\alpha(a)\big) \otimes e_i$$

for all $a \in \mathcal{N}_\phi$ and $v \in D(\gamma^{-\frac{1}{2}})$.

It follows that $(\iota \otimes \omega_{v,w})(V_\phi)\,\Lambda_\phi(a) = \Lambda_\phi\big((\iota \otimes \omega_{\gamma^{-\frac{1}{2}}v,w})\alpha(a)\big)$ for all $a \in \mathcal{N}_\phi$, $v \in D(\gamma^{-\frac{1}{2}})$ and $w \in H_N$.

Using the results of proposition 2.9 of [4], we get that V_ϕ is a unitary element in $B(H_\phi) \otimes N$ such that

1. $(\pi_\phi \otimes \iota)(\alpha(a))V_\phi = V_\phi(\pi_\phi(a) \otimes 1)$ for all $a \in M$.
2. $(\iota \otimes \Delta_N)(V_\phi) = (V_\phi)_{12}(V_\phi)_{13}$

Let ν denote the scaling constant of (N, Δ_N) and define the strictly positive operator P in H_N such that $P^{it}\Lambda_N(a) = \nu^{\frac{t}{2}}\Lambda_N(\tau_t^N(a))$ for all $t \in \mathbb{R}$ and $a \in \mathcal{N}_{\varphi_N}$. We also let $\hat{J}$ denote the modular conjugation of $\hat{\varphi}_N$. Recall that $\hat{J}P\hat{J} = P^{-1}$ (see proposition 2.13(2.7) of [12]) and that $\tau_t^N(x) = P^{it}xP^{-it}$ and $R_N(x) = \hat{J}x^*\hat{J}$ for all $t \in \mathbb{R}$ and $x \in N$.

If γ is a strictly positive operator affiliated with N; arguing as in the proof of proposition 7.5 of [8] lets us conclude that $\tau_t(\gamma) = \gamma$ for all $t \in \mathbb{R}$. So γ and P strongly commute.

If A and B are strictly positive operators in H_N that strongly commute, we denote by $A{\cdot}B$ the closure of the composition AB.

In the next proposition we will rely on relative modular theory (see e.g. section 3.11 of [17]).

Proposition 8.4. *Let $i \in \{1,2\}$. Consider a n.f.s. weight ϕ_i on M with GNS-construction $(H_{\phi_i}, \pi_{\phi_i}, \Lambda_{\phi_i})$ and suppose there exists a positive self-adjoint operator γ_i affiliated with N such that $\Delta_N(\gamma_i) = \gamma_i \otimes \gamma_i$ and for all $a \in \mathcal{M}_{\phi_i}$ and all $v, w \in D(\gamma_i^{\frac{1}{2}})$, we have that the element $(\iota \otimes \omega_{v,w})\alpha(a)$ belongs to $\mathcal{M}_{\phi_i}$ and*

$$\phi_i\big((\iota \otimes \omega_{v,w})\alpha(a)\big) = \phi_i(a)\,\langle \gamma_i^{\frac{1}{2}} v, \gamma_i^{\frac{1}{2}} w \rangle\;.$$

Let ∇ denote the modular operator of ϕ_2,ϕ_1 with respect to the above GNS-constructions. Then

$$V_{\phi_1}\,(\nabla \otimes \gamma_1^{-1} \cdot P^{-1}) = (\nabla \otimes \gamma_2^{-1} \cdot P^{-1})\, V_{\phi_1}\;.$$

Proof. Let T denote the densely defined closed linear map from within H_{ϕ_1} into H_{ϕ_2} such that $\Lambda_{\phi_1}(\mathcal{N}_{\phi_1} \cap \mathcal{N}_{\phi_2}^*)$ is a core for T and $T\Lambda_{\phi_1}(x) = \Lambda_{\phi_2}(x^*)$ for all $x \in \mathcal{N}_{\phi_1} \cap \mathcal{N}_{\phi_2}^*$. Recall that the pair J,∇ is by definition the polar decomposition of T, i.e. $T = J\nabla^{\frac{1}{2}}$.

Choose $v \in D(\gamma_1^{-\frac{1}{2}})$ and $w \in D(\gamma_2^{\frac{1}{2}})$. Let $a \in \mathcal{N}_{\phi_1} \cap \mathcal{N}_{\phi_2}^*$. Then $(\iota \otimes \omega_{\gamma_1^{-\frac{1}{2}} v, w})\alpha(a)$ belongs to $\mathcal{N}_{\phi_1}$ and $\Lambda_{\phi_1}\big((\iota \otimes \omega_{\gamma_1^{-\frac{1}{2}} v, w})\alpha(a)\big) = (\iota \otimes \omega_{v,w})(V_{\phi_1})\Lambda_{\phi_1}(a)$. Moreover, $(\iota \otimes \omega_{\gamma_1^{-\frac{1}{2}} v, w})(\alpha(a))^* = (\iota \otimes \omega_{w, \gamma_1^{-\frac{1}{2}} v})(\alpha(a^*))$ which implies that $(\iota \otimes \omega_{\gamma_1^{-\frac{1}{2}} v, w})(\alpha(a))$ belongs to $\mathcal{N}_{\phi_2}^*$. It follows that $(\iota \otimes \omega_{v,w})(V_{\phi_1})\Lambda_{\phi_1}(a)$ belongs to $D(T)$ and

$$T\big((\iota \otimes \omega_{v,w})(V_{\phi_1})\Lambda_{\phi_1}(a)\big) = \Lambda_{\phi_2}\big((\iota \otimes \omega_{\gamma_1^{-\frac{1}{2}} v, w})(\alpha(a))^*\big) = \Lambda_{\phi_2}\big((\iota \otimes \omega_{w, \gamma_1^{-\frac{1}{2}} v})(\alpha(a^*))\big)$$
$$= (\iota \otimes \omega_{\gamma_2^{\frac{1}{2}} w, \gamma_1^{-\frac{1}{2}} v})(V_{\phi_2})\Lambda_{\phi_2}(a^*) = (\iota \otimes \omega_{\gamma_2^{\frac{1}{2}} w, \gamma_1^{-\frac{1}{2}} v})(V_{\phi_2})\,T\Lambda_{\phi_1}(a)\;.$$

Since such elements $\Lambda_{\phi_1}(a)$ form a core for T, we conclude that

$$(\iota \otimes \omega_{\gamma_2^{\frac{1}{2}} w, \gamma_1^{-\frac{1}{2}} v})(V_{\phi_2})\,T \subseteq T\,(\iota \otimes \omega_{v,w})(V_{\phi_1})\;. \tag{8.21}$$

Taking the adjoint of this inclusion we find that

$$(\iota \otimes \omega_{w,v})(V_{\phi_1}^*)\,T^* \subseteq T^*\,(\iota \otimes \omega_{\gamma_1^{-\frac{1}{2}} v, \gamma_2^{\frac{1}{2}} w})(V_{\phi_2}^*) \tag{8.22}$$

Since $(\iota \otimes \Delta_N)(V_{\phi_i}) = (V_{\phi_i})_{12}\,(V_{\phi_i})_{13}$, we get for every $\omega \in M_*$, that the element $(\omega \otimes \iota)(V_{\phi_i})$ belongs to $D(S_N)$ and $S_N((\omega \otimes \iota)(V_{\phi_i})) = (\omega \otimes \iota)(V_{\phi_i}^*)$. Since $S_N = \tau_{-\frac{i}{2}}^N R_N$, this implies easily that

$$(\iota \otimes \omega_{v,w})(V_{\phi_i}^*) = (\iota \otimes \omega_{\hat{J} P^{\frac{1}{2}} w, \hat{J} P^{-\frac{1}{2}} v})(V_{\phi_i})\;. \tag{8.23}$$

for all $v \in D(P^{-\frac{1}{2}})$ and $w \in D(P^{\frac{1}{2}})$.

So we get for $v \in D(\gamma_1^{\frac{1}{2}} P^{\frac{1}{2}} \gamma_1^{\frac{1}{2}} P^{\frac{1}{2}})$ and $w \in D(\gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}} \gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}})$.

$$(\iota \otimes \omega_{v,w})(V_{\phi_1})\,\nabla = (\iota \otimes \omega_{\hat{J} P^{-\frac{1}{2}} w, \hat{J} P^{\frac{1}{2}} v})(V_{\phi_1}^*)\,T^* T \subseteq T^*\,(\iota \otimes \omega_{\gamma_1^{-\frac{1}{2}} \hat{J} P^{\frac{1}{2}} v, \gamma_2^{\frac{1}{2}} \hat{J} P^{-\frac{1}{2}} w})(V_{\phi_2}^*)\,T$$
$$= T^*\,(\iota \otimes \omega_{\hat{J} \gamma_1^{\frac{1}{2}} P^{\frac{1}{2}} v, \hat{J} \gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}} w})(V_{\phi_2}^*)\,T = T^*\,(\iota \otimes \omega_{\hat{J} P^{\frac{1}{2}} \hat{J} \gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}} w, \hat{J} P^{-\frac{1}{2}} \hat{J} \gamma_1^{\frac{1}{2}} P^{\frac{1}{2}} v})(V_{\phi_2})\,T$$
$$= T^*\,(\iota \otimes \omega_{P^{-\frac{1}{2}} \gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}} w, P^{\frac{1}{2}} \gamma_1^{\frac{1}{2}} P^{\frac{1}{2}} v})(V_{\phi_2})\,T \subseteq T^* T\,(\iota \otimes \omega_{\gamma_1^{\frac{1}{2}} P^{\frac{1}{2}} \gamma_1^{\frac{1}{2}} P^{\frac{1}{2}} v, \gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}} \gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}} w})(V_{\phi_1}),$$

or in other words,

$$(\iota \otimes \omega_{v,w})(V_{\phi_1})\,\nabla \subseteq \nabla\,(\iota \otimes \omega_{\gamma_1^{\frac{1}{2}} P^{\frac{1}{2}} \gamma_1^{\frac{1}{2}} P^{\frac{1}{2}} v, \gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}} \gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}} w})(V_{\phi_1})$$

Since $D(\gamma_1^{\frac{1}{2}} P^{\frac{1}{2}} \gamma_1^{\frac{1}{2}} P^{\frac{1}{2}})$ is a core for $\gamma_1 \cdot P$ and $D(\gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}} \gamma_2^{-\frac{1}{2}} P^{-\frac{1}{2}})$ is a core for $\gamma_2^{-1} \cdot P^{-1}$, it follows easily that

$$(\iota \otimes \omega_{v,w})(V_{\phi_1})\,\nabla \subseteq \nabla\,(\iota \otimes \omega_{(\gamma_1 \cdot P)\, v, (\gamma_2^{-1} \cdot P^{-1})\, w})(V_{\phi_1})$$

for all $v \in D(\gamma_1 \cdot P)$ and $w \in D(\gamma_2^{-1} \cdot P^{-1})$. By lemma 5.9 of [11], we conclude that

$$V_{\phi_1}\,(\nabla \otimes \gamma_1^{-1} \cdot P^{-1}) \subseteq (\nabla \otimes \gamma_2^{-1} \cdot P^{-1})\, V_{\phi_1}\;. \tag{8.24}$$

Taking the adjoint of this equation of this equation and multipliying it with V_{ϕ_1} from the left and the right, we arrive at the other inclusion. $\square$

Corollary 8.5. *Define the strongly continuous one-parameter group κ on N by setting $\kappa_t(x) = \delta_N^{it}\,\tau_t^N(x)\,\delta_N^{-it}$ for all $t \in \mathbb{R}$ and $x \in N$. Then*

$$\alpha\,\sigma_t^{\tilde{\theta}} = (\sigma_t^{\tilde{\theta}} \otimes \kappa_{-t})\alpha$$

for all $t \in \mathbb{R}$.

Proof. Apply the previous proposition with $\phi_1 = \phi_2 = \tilde{\theta}$ and some GNS-construction $(\tilde{H}, \tilde{\pi}, \tilde{\Lambda})$ for $\tilde{\theta}$. In this case , $\gamma_1 = \gamma_2 = \delta_N$ and ∇ is the modular operator for $\tilde{\theta}$ in this GNS-construction. Then we get for $x \in M$ and $t \in \mathbb{R}$ that

$$
\begin{aligned}
(\tilde{\pi} \otimes \iota)(\alpha(\sigma_t^{\tilde{\theta}}(x))) &= V_{\tilde{\theta}}(\tilde{\pi}(\sigma_t^{\tilde{\theta}}(x)) \otimes 1)V_{\tilde{\theta}}^* \\
&= V_{\tilde{\theta}}(\nabla^{it}\tilde{\pi}(x)\nabla^{-it} \otimes 1)V_{\tilde{\theta}}^* = V_{\tilde{\theta}}(\nabla^{it}\tilde{\pi}(x)\nabla^{-it} \otimes \delta_N^{-it}P^{-it}P^{it}\delta_N^{it})V_{\tilde{\theta}}^* \\
&= (\nabla^{it} \otimes \delta_N^{-it}P^{-it})V_{\tilde{\theta}}(\tilde{\pi}(x) \otimes 1)V_{\tilde{\theta}}^*(\nabla^{-it} \otimes P^{it}\delta_N^{it}) \\
&= (\nabla^{it} \otimes \delta_N^{-it}P^{-it})(\tilde{\pi} \otimes \iota)(\alpha(x))(\nabla^{-it} \otimes P^{it}\delta_N^{it}) \\
&= (\tilde{\pi} \otimes \iota)\big((\sigma_t^{\tilde{\theta}} \otimes \kappa_{-t})\alpha(x) \big) \ ,
\end{aligned}
$$

and the corollary follows. $\square$

In the next two proposition we characterize all the weights on M which can be pulled down from Q by the procedure in definition 8.1

Proposition 8.6. *Consider a n.s.f. weight η on Q. Then*

$$\alpha\big((D\tilde{\eta}, D\psi_M)_t\big) = (D\tilde{\eta}, D\psi_M)_t \otimes \delta_N^{-it}$$

for all $t \in \mathbb{R}$.

Proof. Call ∇ the modular operator of the pair $\tilde{\eta}$, ψ_M. By equation (11) of section 3.11 of [17] we know that $(D\tilde{\eta}, D\psi_M)_t = \nabla^{it}\mathbf{\nabla}_M^{-it}$ for all $t \in \mathbb{R}$. Applying the previous proposition to the pair $\tilde{\eta}$, ψ_M (in which case $\gamma_1 = 1$ and $\gamma_2 = \delta_N$), we get that $V_{\psi_M}(\nabla \otimes P^{-1}) = (\nabla \otimes \delta_N^{-1} \cdot P^{-1})V_{\psi_M}$. If we apply the previous proposition to the pair ψ_M, ψ_M (in which case $\gamma_1 = \gamma_2 = 1$, we get that $V_{\psi_M}(\mathbf{\nabla}_M \otimes P^{-1}) \subseteq (\mathbf{\nabla}_M \otimes P^{-1})V_{\psi_M}$.
So we get for all $t \in \mathbb{R}$ that

$$
\begin{aligned}
\alpha\big((D\tilde{\eta}, D\psi_M)_t\big) &= V_{\psi_M}((D\tilde{\eta}, D\psi_M)_t \otimes 1)V_{\psi_M}^* = V_{\psi_M}(\nabla^{it}\mathbf{\nabla}_M^{-it} \otimes P^{-it}P^{it})V_{\psi_M}^* \\
&= (\nabla^{it} \otimes \delta_N^{-it}P^{-it})V_{\psi_M}V_{\psi_M}^*(\mathbf{\nabla}_M^{-it} \otimes P^{it}) = (\nabla^{it} \otimes \delta_N^{-it}P^{-it})(\mathbf{\nabla}_M^{-it} \otimes P^{it}) \\
&= (D\tilde{\eta}, D\psi_M)_t \otimes \delta_N^{-it} \ .
\end{aligned}
$$

$\square$

It is now easy to prove the converse of the previous result.

Proposition 8.7. *Consider a n.s.f. weight ϕ on M such that $\alpha\big((D\phi, D\psi_M)_t\big) = (D\phi, D\psi_M)_t \otimes \delta_N^{-it}$ for all $t \in \mathbb{R}$. Then there exists a unique n.s.f. weight η on Q such that $\phi = \tilde{\eta}$.*

Proof. By the previous proposition, we have for $t \in \mathbb{R}$ that

$$
\begin{aligned}
\alpha((D\phi, D\tilde{\theta})_t) &= \alpha\big((D\phi, D\psi_M)_t(D\psi_M, D\tilde{\theta})_t\big) \\
&= ((D\phi, D\psi_M)_t \otimes \delta_N^{-it})\,((D\psi_M, D\tilde{\theta})_t \otimes \delta_N^{it}) = (D\phi, D\tilde{\theta})_t \otimes 1 \ ,
\end{aligned}
$$

from which we conclude that $(D\phi, D\tilde{\theta})_t$ belongs to Q. By theorem 4.7(1) of [6] we also get that

$$(D\phi, D\tilde{\theta})_{s+t} = (D\phi, D\tilde{\theta})_s \, \sigma_s^{\tilde{\theta}}((D\phi, D\tilde{\theta})_t) = (D\phi, D\tilde{\theta})_s \, \sigma_s^{\theta}((D\phi, D\tilde{\theta})_t) \ .$$

Therefore theorem 5.1 of [17] implies the existence of a n.f.s. weight η on Q such that $(D\eta, D\theta)_t = (D\phi, D\tilde{\theta})_t$ for all $t \in \mathbb{R}$. Theorem 4.7(2) of [6] tells us that $(D\tilde{\eta}, D\tilde{\theta})_t = (D\eta, D\theta)_t = (D\phi, D\tilde{\theta})_t$ for all $t \in \mathbb{R}$. Hence, $\tilde{\eta} = \phi$. $\qquad\square$

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